Finite Element Methods

Four lectures within the minisemester

Numerical Aspects in Applied Mathematics

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Finite element methods (FEM) form a standard tool of modern numerical analysis for the treatment of partial differential equations. These techniques can be considered as specific realization of Ritz-Galerkin obtained by piecewise defined ansatz- and test-functions. However, these piecewise defined functions lead to a large number of new properties and require an adapted generalization of the original problem - called weak formulations.

In this cycle of lectures different aspects of FEM will be studied as follows:

Lecture 1: Discussion of selected topics of functional analysis that are relevant for FEM. This includes elementary properties of Sobolev-spaces as weak derivatives, integration by parts and Green's formula, embedding theorems, Friedrich's inequality. Further, the principle of construction of weak formulations (variational equations) for elliptic boundary value problems is discussed. In particular the inclusion of Neumann- and/or Robin boundary conditions into the variational equation are studied. Abstract existence and uniqueness theorem (Lax-Milgram) and general a-priori estimates (via Cea's lemma) for conforming discretizations are discussed.

Lecture 2: Discretization over triangular and rectangular meshes are considered. Beside common element types like piecewise linear or bilinear C^0-elements also composed elements will be introduced. Bramble-Hilbert lemma is used to estimate interpolation errors and this way to obtain bounds for the convergence order of conforming FEM. Beside stationary (i.e. elliptic) problems also parabolic initial-boundary value problems will be considered.

Lecture 3: Non-conforming FEM, i.e. discretizations that yield discrete problems which do not automatically inherit the essential properties of the underlying variational equation, will be discussed. As specific cases, these methods arise if numerical integration is applied to evaluate numerically integrals that arise in the weak formulation of the boundary value problem or if the chosen discrete space is not contained in the original Sobolev space. Furthermore, the principle of discontinuous Galerkin methods and domain decomposition will be presented.

Lecture 4: Discrete problems generated by FEM are sparse and in practical problems extremely large. Their numerical treatment requires adapted solvers. In the concluding lecture this problem will be addressed by discussing the convergence behavior of simple relaxation methods, multi-grid methods as well as preconditioned conjugate gradient methods. In addition, grid generation techniques are sketched.