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A constructive study of Markov equilibria in stochastic games with strategic complementarities ☆

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Abstract

We study a class of infinite horizon, discounted stochastic games with strategic complementarities. In our class of games, we prove the existence of a stationary Markov Nash equilibrium, as well as provide methods for constructing this least and greatest equilibrium via a simple successive approximation schemes. We also provide results on computable equilibrium comparative statics relative to ordered perturbations of the space of games. Under stronger assumptions, we prove the stationary Markov Nash equilibrium values form a complete lattice, with least and greatest equilibrium value functions being the uniform limit of approximations starting from pointwise lower and upper bounds.

JEL classification: C62; C73

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1. Introduction and related literature

Since the class of infinite horizon discounted stochastic games was first introduced by Shapley [44], the question of existence and characterization of equilibrium has been the object of extensive study.[1] The focal point of a great deal of the recent work on equilibrium in stochastic games has been on sufficient conditions for the existence of minimal state space Markov stationary Nash equilibrium (MSNE).[2] Moreover, as stochastic games have become a fundamental tool for studying strategic interactions in dynamic economic models (and, in particular, in models where agents possess some form of limited commitment over time), these questions have become particularly important to answer. Indeed, work using stochastic games has arisen in such diverse fields in economics as: (i) equilibrium models of stochastic growth without commitment [6,12], (ii) international lending and sovereign debt [10], (iii) optimal Ramsey taxation [41], (iv) models of savings and asset prices with hyperbolic discounting [24], (v) dynamic political economy [30], (vi) dynamic negotiations with status quo [18], or (vii) dynamic oligopoly models [14], among others.

Apart from the question of equilibrium existence and its characterization, the computation of MSNE has also become a central focus of applied researchers. For example, when applying calibration or estimation techniques to provide characterizations of dynamic equilibrium via approximate solutions, a prerequisite for the rigorous implementation of such numerical methods is to have access to a sharp set of theoretical tools that first characterize the structure of elements of the set of MSNE in the economy under study at a fixed set of parameters, and then identify how the equilibrium set varies in deep parameters of the game. When equilibrium is unique, such questions can be address in somewhat obvious ways; but when multiplicities of equilibria are present, these questions become more subtle. In particular, in the presence of multiple equilibria, when considering numerical approaches to the characterization question, it proves particularly powerful to have access to *constructive* fixed point methods that are "order stable" in deep parameters, where the theoretical methods used to study the MSNE structure can be tied directly to the implementations of numerical methods. For finite games, the question of existence, characterization, and computation of MSNE has been essentially resolved.[3,4] Unfortunately, for infinite games, although the equilibrium existence question has received a great deal of attention, results that provide sharp characterizations of the MSNE set, as well as how it varies in deep parameters are needed. This is true not only to address the question of accuracy of approximation methods, but also to develop notions of qualitative and quantitative stability.[5]

The aim of this paper is to address all of these issues (i.e., existence, characterization, and computation of equilibrium comparative statics) within the context of a single unified

[1] See [42] or [36] for an extensive survey of results, along with references, as well as [17,31] for more recent discussions.

[2] In this paper, we study both the construction of Markov stationary Nash equilibrium strategies (MSNE) and their associated values (MSNE values). The pair (MSNE, MSNE value) is what we refer to as a SMNE.

[3] By "finite game" we mean a dynamic/stochastic game with a (a) finite horizon and (b) finite state and strategy space game. By an "infinite game", we mean a game where either (c) the horizon is countable (but not finite), or (d) the action/state spaces are uncountable. We shall focus on stochastic games where both (c) and (d) are present.

[4] For example, for work that discusses existence questions, see [21]; for work that studies computational issues, see [25]; finally, for work that discusses estimation issues, see [2,40] or [39].

[5] There are exceptions to this remark. For example, in [11], a truncation argument for constructing a SMNE in symmetric games of capital accumulation is proposed. In general, though, in this literature, a unified approach to approximation and existence has not been addressed.

methodological approach for MSNE. Our methods are *constructive* and *monotone*, where in our work, monotonicity is relative to pointwise partial orders on function spaces for *both* equilibrium values and pure strategies. What is important in our work is that is some cases, our monotone methods can be applied to characterize *non-monotone* Markovian equilibrium. To obtain sufficient conditions for our constructive monotone methods to be applied relative to the set of MSNE, we study an important subclass of stochastic games, namely these with strategic complementarities and positive externalities. This class of games is important among others, as equilibrium is known to exists relative to general assumptions, but also as many other classes of games are simply not computable.

For infinite horizon stochastic supermodular games, we prove a number of new results relative to the existing literature (e.g., [16,9] or [37]). First, we prove the existence of MSNE in broader spaces of (bounded, measurable) pure strategies. Second, and perhaps most importantly per applications, we develop reasonable sufficient conditions for MSNE to exist over *very general state spaces*. Third, we give sufficient conditions under which the set of MSNE values forms a complete lattice of Lipschitz continuous functions. Fourth, we contribute to the literature that studies specific forms of transition kernels, and we are able to show the full power of the mixing assumption studied extensively by Nowak and coauthors in a class of stochastic games. Finally, we prove our results using minimal assumptions (i.e. we are also able to present counterexamples that violate both our assumptions and results).

Along these lines, our results contribute to the recent literature on the non-existence of MSNE in the class of discounted stochastic games with absolute continuity conditions (ACC) (see [31]) or with additional noise (see [17]). More specifically, we provide results on the existence of MSNE in a stochastic game over an uncountable state space, where transition between states is *not* absolutely continuous. This result complements the recent result of Levy [31] concerning the importance of ACC; namely, that this condition is neither sufficient nor necessary for the existence of MSNE. Moreover, we obtain our existence results per MSNE in pure strategies *without* introducing additional correlation or noise as is done in the work of Nowak and Raghavan [38] or Duggan [17].

Perhaps most importantly, unlike the existing work in this large and emerging literature, our methods provide a *single unified* approach to both finite and infinite horizon games. That is, we give conditions under which infinite horizon MSNE are simply the limits of equilibria in truncated finite horizon stochastic games. This fact is particularly important for implementations of numerical methods, where truncation arguments play a key role in numerical computation of MSNE. Further, it means our results can be viewed as a direct generalization of those obtained in [7] for ordered optimal solutions for discounted dynamic lattice programming models to the setting of multiagent decision theory and dynamic equilibria in a general class of stochastic supermodular games. Finally, our results give conditions, where computable comparative statics/dynamics is available for MSNE in the infinite horizon stochastic game. Such results are particularly important, when one seeks to construct a stable selections relative to the set of MSNE that are numerically (and theoretically) tractable as functions of the deep parameters of the economy/game.

The rest of the paper is organized as follows. Section 2 presents a motivating example that highlights the nature and applicability of our results. Then Section 3 describes our class of stochastic supermodular games, and states the formal definition of MSNE. We then show in Section 3.2 for some minimal sets of sufficient conditions, MSNE exists, and extremal equilibrium values and pure strategies can be computed via successive approximations. In Section 3.4, we present a set of equilibrium comparative statics/dynamics results. Finally, in Section 4, we present three applications of our results.

2. Motivating example

We start with a simple motivating example for our results relative to the question of computing extremal MSNE in a two player infinite horizon stochastic game with uncountable and two-dimensional state space. This example, in particular, highlights some of the essential issues and results raised in this paper (as we shall note in a moment). Each period a pair of states $(s_1, s_2) \in [0, 1] \times [0, 1] =: S$ is drawn and after observing it players choose one of two actions 1 or 0. The payoffs from the stage game are given in the following table:

	1	0
1	s_1, s_2	$s_1 - c, b$
0	$b, s_2 - c$	$0, 0$

This game, for example, can be thought of as a stylized partnership game in which players choose to keep putting effort into their partnership (cooperate) or to quit. That payoffs/states s_1, s_2 represent the returns to each player from putting effort into the partnership. Parameter c represents the losses associated with staying in the partnership, when the other player walks out; parameter b represents a potential benefit from cheating on a cooperating partner. Assume $1 > c > b > 0$. Then, this stage game is clearly a supermodular game with positive externalities. For such a one shot game, we have the following pure strategy Nash equilibria depending on parameters s_1, s_2:

s_1/s_2	$< b$	$\in (b, c)$	$> c$
$< b$	$(0, 0)$	$(0, 0)$	$(0, 1)$
$\in (b, c)$	$(0, 0)$	two NE $(1, 1)$ and $(0, 0)$	$(1, 1)$
$> c$	$(1, 0)$	$(1, 1)$	$(1, 1)$

Let state $s = (s_1, s_2)$ be drawn from distribution $Q(\cdot|s, a)$ parameterized by current action $a = (a_1, a_2)$. Now, say the stochastic transition of state s is given by a transition kernel $Q(\cdot|s, a) = g(s, a)\lambda(\cdot|s) + (1 - g(s, a))\delta_0(\cdot)$, where $\lambda(\cdot|s)$ is a measure on S and δ_0 is a Delta Dirac concentrated at $(0, 0) \in S$. In this case, the main results of this paper show there exists the greatest and the least MSNE (see Theorem 3.1), which also can be used from a numerical viewpoint to bound the set of *all* MSNE (see Lemma 3.1). Moreover, our theorems show we can develop a simple successive approximation scheme to compute extremal MSNE (see Corollary 3.1).

To see how easy it is to apply our computational techniques to this game, we now specify some parameters for the game, and compute extremal MSNE under those parameter settings. For example, let $c = .8$, $b = .2$, discount factor $\beta = .9$ and assume that the function in the above specification of Q is given by $g(s, a) = \frac{(s_1+s_2)}{2} \frac{(a_1+a_2)^2}{4}$, with λ uniformly distributed on $[0, 1] \times [0, 1]$. In Fig. 1, we present the results of the computations of the greatest and the least expected values (as well as the iterations to these values): $\int_S v(s)\lambda(ds)$. Having computed the expected values, we can also construct both the greatest and the least MSNE (see Fig. 2). We should mention, apart from proving equilibrium existence and computation results, later in the paper we shall prove results on equilibrium monotone comparative statics (see Theorem 3.4), and these results on how to compare equilibria can be easily computed for different parameter settings for this game (see the discussion of these results in Section 3.3).

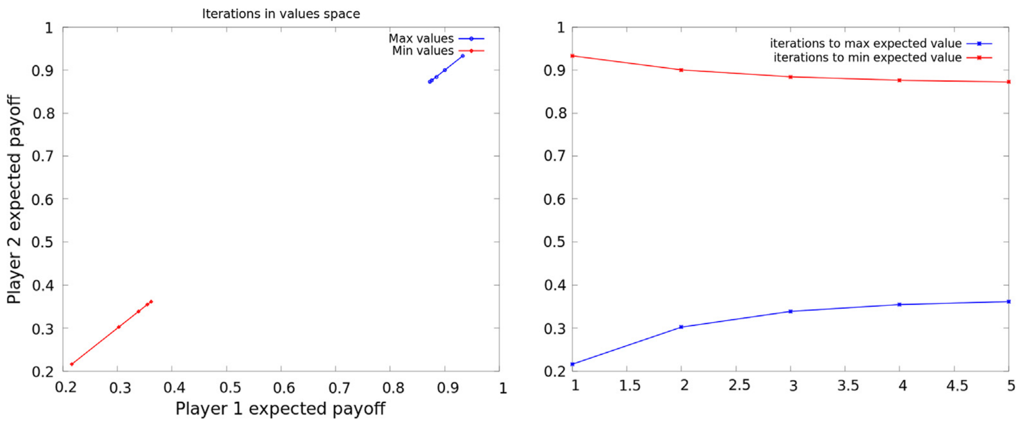

Fig. 1. Convergence of iterations (expected values) from above and below to extremal MSNE expected values. Iterations in expected value space and MSNE value bounds (left panel); speed of convergence (right panel).

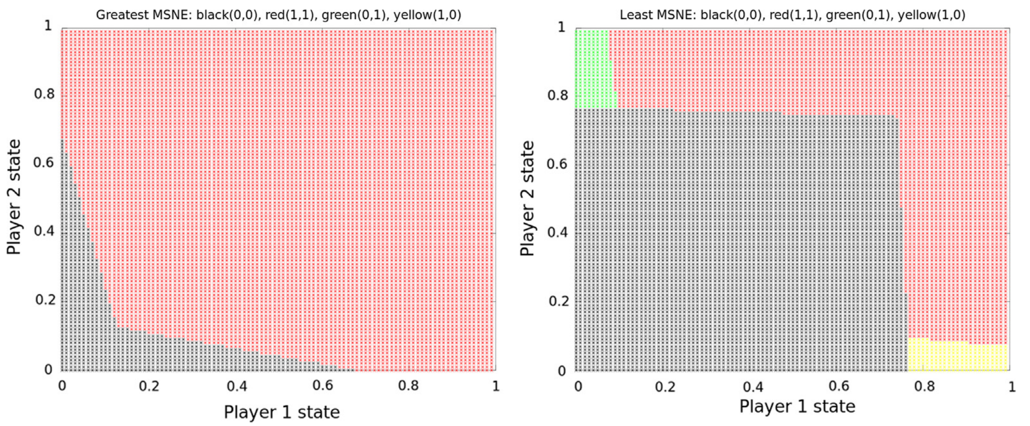

Fig. 2. Greatest and least MSNE. Colored regions denote parts of the state space where a particular profile of actions is a MSNE: black: $(0,0)$, green: $(0,1)$, yellow: $(1,0)$, red: $(1,1)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Now, notice using our methods, this game is quite simple to compute. However, also notice that the celebrated APS correspondence-based approach, first suggested in the seminal work of Abreu, Pearce and Stacchetti [1], but later used to verify the existence of equilibrium in stochastic games (e.g., see [15] or [33]), *cannot* be applied to analyze the MSNE of this game. This is true for at least two reasons. First, APS type methods fail in the case of stochastic games with *uncountable* two-dimensional state spaces (as conditions that guarantee the existence of a nonempty, compact set of equilibrium values for multidimensional state space are not known). Second, APS type methods focus on the sequential or long-memory/nonstationary Markov equilibria, and have little to say about the structure of (short-memory) MSNE. Finally, it bears mentioning that other topological techniques exploiting continuity conditions also fail here as the transition probability Q need *not* satisfy the continuity conditions needed to obtain sufficient conditions for existence (see [31] for an excellent discussion).

3. Main results

3.1. Definitions and assumptions

Consider an n-player discounted infinite horizon stochastic game in discrete time. The primitives of our class of games are given by the tuple $\{S, (A_i, \tilde{A}_i, \beta_i, u_i)_{i=1}^n, Q, s_0\}$, where $S = [0, \bar{S}] \subset \mathbb{R}^k$ is the state space, $A_i \subset \mathbb{R}^{k_i}$ player i action space with $A = \times_i A_i$, β_i is the discount factor for player i, $u_i : S \times A \to \mathbb{R}$ is the one-period payoff function, and $s_0 \in S$ the initial state of the game. For each $s \in S$, the set of feasible actions for player i is given by $\tilde{A}_i(s)$, which is assumed to be compact Euclidean interval in $\mathbb{R}^{k_i}$. By Q, we denote a transition function that specifies for any current state $s \in S$ and current action $a \in A$, a probability distribution over the realizations of next period states $s' \in S$.

Using this notation, we can provide a formal definition of a (Markov, stationary) strategy, payoff, and a Nash equilibrium. A *strategy* for a player i is denoted by $\Gamma_i = (\gamma_i^1, \gamma_i^2, \ldots)$, where γ_i^t specifies an action to be taken at stage t as a function of history of all states s^t, as well as actions a^{t-1} taken till stage $t-1$ of the game. If a strategy depends on a partition of histories limited to the current state s_t, then the resulting strategy is referred to as *Markov*. If for all stages t, we have a Markov strategy given as $\gamma_i^t = \gamma_i$, then strategy Γ_i for player i is called a *Markov stationary strategy*, and denoted simply by γ_i.

For a strategy profile $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$, and initial state $s_0 \in S$, the expected payoff for player i can be denoted by:

$$U_i(\Gamma, s_0) = (1 - \beta_i) \sum_{t=0}^{\infty} \beta_i^t \int u_i(s_t, a_t)\, dm_i^t(\Gamma, s_0),$$

where m_i^t is the stage t marginal on A_i of the unique probability distribution induced on the space of all histories for Γ, given by Ionescu–Tulcea's theorem. A Markov stationary strategy profile $\Gamma^* = (\Gamma_i^*, \Gamma_{-i}^*)$ is then a *Markov stationary Nash equilibrium* (*MSNE*) if and only if Γ^* is feasible, and for any i, and all feasible Γ_i, we have

$$U_i\big(\Gamma_i^*, \Gamma_{-i}^*, s_0\big) \geqslant U_i\big(\Gamma_i, \Gamma_{-i}^*, s_0\big).$$

We now state some initial conditions on the primitives of the game that will be required for our methods to work.

Assumption 1 *(Preferences). For $i = 1, \ldots, n$ let*:

- *u_i be continuous on A and measurable on S, with $0 \leqslant u_i(s, a) \leqslant \bar{u}$,*
- $(\forall a \in A) u_i(0, a) = 0$,
- *u_i be increasing in a_{-i},*
- *u_i be supermodular in a_i for each (a_{-i}, s), and has increasing differences in $(a_i; a_{-i})$,*
- *for all $s \in S$ the sets $\tilde{A}_i(s)$ be nonempty, compact intervals and $s \to \tilde{A}_i(s)$ be a measurable correspondence.*

Assumption 2 *(Transition). Let Q be given by*:

- *$Q(\cdot|s, a) = g_0(s, a)\delta_0(\cdot) + \sum_{j=1}^{L} g_j(s, a)\lambda_j(\cdot|s)$, where*

- *for* $j = 1, \ldots, L$, *the function* $g_j : S \times A \to [0, 1]$ *is continuous on* A, *measurable on* S, *increasing and supermodular in* a *for fixed* s, $g_j(0, a) = 0$ *with* $\sum_{j=1}^{L} g_j(\cdot) + g_0(\cdot) \equiv 1$,
- $(\forall s \in S, j = 1, \ldots, L)$, $\lambda_j(\cdot|s)$ *is a Borel transition probability on* S,[6]
- δ_0 *is a probability measure concentrated at point* 0.

3.2. Existence and computation of SMNE

Let $Bor(S, \mathbb{R}^n)$ denote the set of Borel measurable functions from S into $\mathbb{R}^n$, and consider its following subset:

$$\mathcal{B}^n(S) := \{v \in Bor(S, \mathbb{R}^n) \colon \forall_i v_i(0) = 0, \ \|v_i\| \leqslant \bar{u}\}.$$

Equip the space $Bor(S, \mathbb{R}^n)$ with its pointwise partial order, and the subset $\mathcal{B}^n(S)$ with its relative partial order. We are now prepared to state the first main theorem of this section, which concerns the existence and approximation of MSNE.

Theorem 3.1 *(Existence and approximation of SMNE). Let Assumptions 1 and 2 be satisfied. Then*

- *there exists the greatest* (ψ^*) *and the least* (ϕ^*) *MSNE in* $\mathcal{B}^n(S)$, *with associated greatest MSNE value* (w^*) *and least MSNE value* (v^*),
- *if* γ^* *is an arbitrary MSNE, then* $(\forall s \in S)$, *we have the equilibrium bounds* $\phi^*(s) \leqslant \gamma^*(s) \leqslant \psi^*(s)$. *Further, if* μ^* *is equilibrium payoff associated with any stationary Markov Nash equilibrium* γ^*, *then* $(\forall s \in S)$, *we have the MSNE value bounds given by:* $v^*(s) \leqslant \mu^*(s) \leqslant w^*(s)$,
- *both greatest and least MSNE, as well as their associated MSNE values, can be pointwise approximated as limits of monotone sequences.*

To provide some insight into our construction of MSNE, first for a vector of continuation values $v = (v_1, v_2, \ldots, v_n) \in \mathcal{B}^n(S)$, consider the auxiliary one-period, n-player game G_v^s with action sets $\tilde{A}_i(s)$ and payoffs given as follows:

$$\Pi_i(v_i, s, a_i, a_{-i}) := (1 - \beta_i)u_i(s, a_i, a_{-i}) + \beta_i \int_S v_i(s') Q(ds' | s, a_i, a_{-i}).$$

Note, under Assumptions 1 and 2, the auxiliary game G_v^s is a supermodular game for any (v, s). Therefore, it possesses a greatest $\bar{a}(s, v)$ and least $\underline{a}(s, v)$ (measurable[7]) pure strategy Nash equilibrium (e.g., [48] and [50]), as well as corresponding greatest $\bar{\Pi}^*(v, s)$ and least $\underline{\Pi}^*(v, s)$ MSNE values, where the equilibrium payoffs are given as $\Pi^*(v, s) = (\Pi_1^*(v, s), \Pi_2^*(v, s), \ldots, \Pi_n^*(v, s))$.

From these equilibrium payoffs, define a pair of extremal value function operators

$$\bar{T}(v)(s) = \bar{\Pi}^*(v, s) \quad \text{and} \quad \underline{T}(v)(s) = \underline{\Pi}^*(v, s),$$

and let $T^j(v)$ denote the j-iteration/orbit of the operator T from the initial function v. Then, we can generate recursively a sequence of lower (resp., upper) bounds for equilibrium values $\{v^j\}_{j=0}^{\infty}$

[6] This means among other that function $s \to \int_S v(s')\lambda_j(ds'|s)$ is measurable for any integrable v.

[7] Lemma 5.3 shows that both extremal equilibria and their corresponding values are measurable.

(resp., $\{w^j\}_{j=0}^{\infty}$) where $v^{j+1} = \underline{T}(v^j)$ for $j \geqslant 1$ from the initial guess $v^0(s) = (0, 0, \ldots, 0)$ (resp., $w^{j+1} = \bar{T}(w^j)$ from initial guess $w^0(s) = (\bar{u}, \bar{u}, \ldots, \bar{u})$ for $s > 0$ and $w^0(0) = 0$). Notice, for both lower (resp., upper) value iterations, we can associate sequences of pure strategy Nash equilibrium strategies $\{\phi^j\}_{j=0}^{\infty}$ (resp., $\{\psi^j\}_{j=0}^{\infty}$), which are defined by the recursion $\phi^j = \underline{a}(s, v^j)$ (resp., $\psi^j = \bar{a}(s, w^j)$).

Then, the existence and computation of MSNE is reduced to studying the limiting properties of this collection of iterative processes. We have the following key result.

Lemma 3.1 *(The successive approximation of SMNE). Under Assumptions 1 and 2 we have*

1. *(for fixed $s \in S$), $\phi^j(s)$ and $v^j(s)$ are increasing sequences and $\psi^j(s)$ and $w^j(s)$ are decreasing sequences,*
2. *for all j we have $\phi^j \leqslant \psi^j$ and $v^j \leqslant w^j$ (pointwise),*
3. *the following limits exist: $(\forall s \in S) \lim_{j\to\infty} \phi^j(s) = \phi^*(s)$ and $(\forall s \in S) \lim_{j\to\infty} \psi^j(s) = \psi^*(s)$,*
4. *the following limits exist: $(\forall s \in S) \lim_{j\to\infty} v^j(s) = v^*(s)$ and $(\forall s \in S) \lim_{j\to\infty} w^j(s) = w^*(s)$,*
5. *ϕ^* and ψ^* are stationary Markov Nash equilibria in the infinite horizon stochastic game. Moreover, v^* and w^* are equilibria payoffs associated with ϕ^* and ψ^* respectively.*

As our existence result in Theorem 3.1 is obtained under different assumptions than those found in the existing literature for stochastic supermodular games (e.g., [16,9] or [37]), it is useful to provide a brief discussion of the central differences between our results and those found in the existing literature.

First, relative to Curtat (see also [8]), we do not require the payoffs or the transition probabilities to be Lipschitz continuous, nor do we impose stochastic equicontinuity conditions (which are used in both of those papers to obtain existence in the infinite horizon game).[8] These conditions appear strong relative to many economic applications.[9] Further, we also do not impose any conditions on payoffs and stochastic transitions that imply "double increasing differences" in payoff structures in the sense of Granot and Veinott [23], nor do we impose strong concavity conditions such as strict diagonal dominance to obtain our existence results.[10] Also, as compared to [9], we do not require the class of games to have a *single* dimensional state space.[11]

[8] Also, relative to Amir's case, as we do not require *any* form of continuity of λ_j with respect to the state s, we do not satisfy Amir's assumption T1. Further, although we both require that the stochastic transition structure Q is stochastically supermodular with a, we do not require increasing differences with (a, s) as Amir does, nor do we require the stochastic monotonicity conditions for Q in s.

[9] For example, such conditions rule out payoffs that are consistent with Inada type assumptions (e.g., Cobb–Douglas utility).

[10] Both of these sets of assumptions (e.g., double increasing differences and strong diagonal dominance) are required by both of these authors for existence. That is, they each need to obtain unique Lipschitz Nash equilibrium in the stage game that is *continuous* with continuation v for their eventual application of a topological fixed point theorem per existence. Similar NE uniqueness conditions are required by Nowak [37]. The difference with our setup relative to theirs is of utmost importance. Specifically, without equilibrium uniqueness in the stage game with continuation v, these authors cannot construct an upper-hemicontinuous correspondence, a condition necessary to apply Fan–Glicksberg fixed point theorem. Similar issues arise when trying to apply Schauder's theorem (e.g. as in [16]).

[11] Also, as opposed to [9], our feasible action correspondences $\tilde{A}_i$, and payoff/transition structures u_i and g_j are only required to be measurable with s, as opposed to upper semicontinuous as in [9].

More generally, and equally critical when comparing our results to those in the existing literature, we do not assume *any increasing differences* between actions and states. This last difference is also critical when comparing our results to those in [9]. That is, we do not require *monotone Markov equilibrium* to obtain existence; rather, we just need enough complementarity to construct *monotone operators*. Therefore, our sufficient conditions are able to distinguish between the role of monotonicity conditions needed for the existence and computation of MSNE (e.g., to obtain monotone operators in sufficiently chain complete partially ordered sets) from those conditions needed for the existence of monotone MSNE.

Second, to obtain our results, we do need to impose a very important condition on the stochastic transitions Q which are *stronger* than those needed for existence in the work of Curtat and Amir. In particular, we assume the transition structure induced by Q can be represented as a convex combination of $L+1$ probability measures, of which one measure is a delta Dirac concentrated at 0. As a result, with probability g_0, we set the next period state to zero; with probability g_j, the distribution is drawn from the non-degenerate distribution λ_j (where, in this latter case, this distribution does not depend on the vector of actions a, but is allowed to depend on the current state s). Also, although we assume each λ_j is stochastically ordered relative to the Dirac delta δ_0, we do not impose stochastic orders among the various measures λ_j. This "mixing" assumption for transition probabilities has been discussed extensively in the literature. Surprisingly, the main strength of this assumption has not been fully used (see [37]) until the work of Balbus, Reffett and Woźny [13] in the context of paternalistic altruism economies, as well as this present paper. Clearly, the restrictive part of our assumption is that we require existence of an *absorbing state* 0 that gives the minimal value for any $v \in \mathcal{B}^n(S)$. This is required by our techniques as we need to show that operators $\bar{T}, \underline{T}$ are well-defined transformations of $\mathcal{B}^n(S)$ (and hence, require $\bar{T}(v)(0) = \underline{T}(v)(0) = 0$). This latter assumption, although it can be potentially restrictive in some applications, still allows for some generalizations. For example, our assumptions can be easily generalized to allow any absorbing state $\underline{s} \in S$, such that $v(\underline{s}) = z$, where $z = \min_{s \in S} v(s)$ for any $v \in \mathcal{B}^n(S)$, and the unique Nash equilibrium value in the auxiliary game G_v^s has value z at $\underline{s}$, for any integrable continuation v. Moreover, $\underline{s}$ need not be minimal in S unless v is monotone. Also, we can allow for other absorbing states (and hence, the probability of reaching 0 can be reduced to *zero* (see Theorem 3.5)).

3.3. Uniform error bounds for Lipschitz continuous SMNE

We now turn to error bounds for approximate solutions. We initially discuss two sets of results in this section. First, the limits of iterations used to compute extremal solutions in Theorem 3.1 and Lemma 3.1 are only relative to *pointwise convergence*. With slightly stronger assumptions, we can obtain those limits in *uniform convergence*. Second, to obtain uniform error bounds, we need to make some stronger assumptions on the primitives which will allow us to address the question of Lipschitz continuity of equilibrium strategies. The assumptions we will use are common in applications (e.g., compare the assumptions we impose to those in [16]).

In this section, we assume that the state space S is endowed with a *taxi-norm* $\|\cdot\|_1$.[12] The spaces A_i and A are endowed with the sup-norm. We say a function $f : S \to A$ is *M-Lipschitz continuous* if and only if for all $i = 1, \ldots, n$, $\|f_i(x) - f_i(y)\| \leqslant M\|x - y\|_1$. Note, if f_i is differentiable, then M-Lipschitz continuity is equivalent to that each partial derivative being

[12] Taxi-norm of vector $x = (x_1, \ldots, x_k)$ is defined as $\|x\|_1 = \sum_{i=1}^{k} |x_i|$.

uniformly bounded above by M. To obtain uniform convergence and uniform approximation results, we will need some additional structure on the primitives of the game, that is discussed in the following assumption.

Assumption 3. *For all i, j:*

- *u_i, g_j are twice continuously differentiable on an open set containing $S \times A$,*[13]
- *u_i is increasing in (s, a_{-i}) and satisfies cardinal complementarity*[14] *in a_i and (a_{-i}, s),*
- *u_i satisfies a strict dominant diagonal condition in a_i, a_{-i} for fixed $s \in S, s > 0$, i.e. if we denote $a_i \in \mathbb{R}^{k_i}$ as $a_i := (a_i^1, \dots, a_i^{k_i})$, then*

$$\forall_{i=1,\dots,n} \forall_{j=1,\dots,k_i} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{k_\alpha} \frac{\partial^2 u_i}{\partial a_i^j \partial a_\alpha^\beta} < 0,$$

- *g_j is increasing in (s, a) and has cardinal complementarity in a_i and (s, a_{-i}),*
- *g_j satisfies a strict dominant diagonal condition in a_i, a_{-i} for fixed $s \in S, s > 0$,*
- *λ_j has a Feller property,*
- *for each increasing, Lipschitz and bounded by $\bar{u}$ function f, the function $\eta_j^f(s) := \int_S f(s')\lambda_j(ds'|s)$ is increasing and Lipschitz continuous with a constant $\bar{\eta}$,*[15]
- *$\tilde{A}_i(s) := [0, \tilde{a}_i(s)]$ and each function $s \to \tilde{a}_i(s)$ is Lipschitz continuous and isotone function.*[16]

Define the set CM^N of N-tuples of increasing Lipschitz continuous functions with some constant M on S. Equip CM^N with a partial order, i.e.: for $w, v \in CM^N$, where $w = (w_1, \dots, w_N)$ and similarly for v, let

$$w \geqslant v \quad \text{iff} \quad (\forall i = 1, \dots, N)(\forall s \in S) w_i(s) \geqslant v_i(s).$$

Clearly, CM^N is a complete lattice. It is also nonempty, convex and compact in the sup-norm.

It is also important to note that CM^N is closely related to the space where equilibrium is constructed in [16]. We should note, though, that there are two key differences between our game and those studied in Curtat. First, we allow the choice set A_i to depend on s (where, he assumes A_i independent of s). Second, under our assumptions, the auxiliary game G_v^0 has a continuum of Nash equilibria, and hence we need to close the Nash equilibrium correspondence in state 0. These technical differences are addressed in Lemma 5.4 and proof of the next theorem.

Before we proceed, we provide a few additional definitions. For each twice continuously differentiable function $f : A \to \mathbb{R}$, we define the following mappings:

$$\mathcal{L}_{i,j}(f) := -\sum_{\alpha=1}^{n} \sum_{\beta=1}^{k_\alpha} \frac{\partial^2 f}{\partial a_i^j \partial a_\alpha^\beta}, \qquad U_{i,j,l}^2 := \sup_{s \in S, a \in \tilde{A}(s)} \frac{\partial^2 u_i}{\partial a_i^j \partial s_l}(s, a),$$

[13] Note that this implies that u_i and g_j are bounded Lipschitz continuous functions on compact $S \times A$.

[14] That is, the payoffs are supermodular in a_i and have increasing differences in (a_i, s) and in (a_i, a_{-i}).

[15] This condition is satisfied if each of measures $\lambda_j(ds'|s)$ has a density $\rho_j(s'|s)$ and the function $s \to \rho_j(s'|s)$ is Lipschitz continuous uniformly in s'.

[16] Each coordinate $\tilde{a}_i^j$ is Lipschitz continuous. Notice, this implies that the feasible actions are Veinott strong set order isotone.

$$G^1_{i,j} := \sup_{s\in S, a\in\tilde{A}(s)} \sum_{\alpha=1}^{L} \frac{\partial g_\alpha}{\partial a_i^j}(s,a), \qquad G^2_{i,j,l} := \sup_{s\in S, a\in\tilde{A}(s)} \sum_{\alpha=1}^{L} \frac{\partial^2 g_\alpha}{\partial a_i^j \partial s_l}(s,a),$$

$$M_0 := \max\left\{ \frac{(1-\beta_i)U^2_{i,j,l} + \beta_i \bar{\eta} G^1_{i,j} + \beta_i \bar{u} G^2_{i,j,l}}{-(1-\beta_i)\mathcal{L}_{i,j}(u_i)}: \right.$$
$$\left. i = 1,\dots,n,\ j = 1,\dots,k_i,\ l = 1,\dots,k \right\}.$$

With these definitions in mind, we can now prove our main result on existence of Lipschitz continuous MSNE:

Theorem 3.2 *(Lipschitz continuity). Let Assumptions 1, 2, 3 be satisfied. Assume additionally that each $\tilde{a}_i(\cdot)$ is Lipschitz continuous with a constant less then M_0. Then, stationary Markov Nash equilibria ϕ^*, ψ^* and corresponding values v^*, w^* are all Lipschitz continuous.*

We can now study the uniform approximation of MSNE. Our results appeal to a version of Amann's theorem (e.g., [4, Theorem 6.1]). For this argument, denote by $\mathbf{0}$ (by $\bar{\mathbf{u}}$ respectively) the n-tuple of function identically equal to 0 ($\bar{u}$ respectively) for all $s > 0$. Observe, under Assumption 3, the auxiliary game has a unique NE value, so we have

$$\underline{T}(v) = \bar{T}(v) := T(v).$$

Our result then is as follows:

Corollary 3.1 *(Uniform approximation of extremal SMNE). Let Assumptions 1, 2 and 3 be satisfied. Then $\lim_{j\to\infty} \|T^j \mathbf{0} - v^*\| = 0$ and $\lim_{j\to\infty} \|T^j \bar{\mathbf{u}} - w^*\| = 0$, with $\lim_{j\to\infty} \|\phi^j - \phi^*\| = 0$ and $\lim_{j\to\infty} \|\psi^j - \psi^*\| = 0$.*

Notice, the above corollary assures that the convergence in Theorem 3.1 is *uniform*. We also obtain a stronger characterization of the set of MSNE in this case, namely, the set of MSNE equilibrium value functions form a complete lattice.

Theorem 3.3 *(Complete lattice structure of SMNE value set). Under Assumptions 1, 2 and 3, there exist $M > 0$ such that, the set of Markov stationary Nash equilibrium values in CM^n is a nonempty complete lattice.*

The above result provides a further characterization of a MSNE strategies, as well as their corresponding set of equilibrium value functions. From a computational point of view, not only are the extremal values and strategies Lipschitzian (as known from previous work), they can also can be uniformly approximated by a simple algorithm. Also, note, it is not clear if the set of MSNE in $CM^{\sum_i k_i}$ is necessarily a complete lattice.

3.4. Monotone comparative dynamics

We next study the question of sufficient conditions under which our games exhibit equilibrium monotone comparative statics relative to both extremal fixed point values v^*, w^*, as well as the corresponding pure strategy extremal equilibria ϕ^*, ψ^*. We also consider the related question of ordered equilibrium stochastic dynamics.

Along these lines, we first parameterize our stochastic game by a set of parameters $\theta \in \Theta$, where Θ is some partially ordered set. One way to interpret θ is a vector whose elements include parameters representing ordered perturbations to any of the following primitive data of the game: (i) period payoffs u_i, (ii) the stochastic transitions g_j and λ_j, and (iii) feasibility correspondence A_i. Alternatively, we can think of elements θ as being policy parameters of the environment governing the setting of taxes or subsidies (as, for example, in a dynamic policy game with strategic complementarities).

So, consider parameterized versions of Assumptions 1 and 2 as follows:

Assumption 4 *(Parameterized preferences). For $i = 1, \dots, n$ let*:

- *$u_i : S \times A \times \Theta \to \mathbb{R}$ be a function and $u_i(\cdot, s, \theta)$ continuous on A for any $s \in S$, $\theta \in \Theta$ with $u_i(\cdot) \leqslant \bar{u}$, and $u_i(\cdot, \cdot, \theta)$ is measurable for all θ,*
- *$(\forall a \in A, \theta \in \Theta)\ u_i(0, a, \theta) = 0$,*
- *u_i be increasing in (s, a_{-i}, θ),*
- *u_i be supermodular in a_i for fixed (a_{-i}, s, θ), and has increasing differences in $(a_i; a_{-i}, s, \theta)$,*
- *for all $s \in S$, $\theta \in \Theta$, the sets $\tilde{A}_i(s, \theta)$ are nonempty, measurable (for given θ), compact intervals and $\tilde{A}_i$ measurable multifunction that is both ascending in the Veinott's strong set order,[17] and expanding under set inclusion[18] with $\tilde{A}_i(0, \theta) = 0$.*

Assumption 5 *(Parameterized transition). Let Q be given by*:

- *$Q(\cdot|s, a, \theta) = g_0(s, a, \theta)\delta_0(\cdot) + \sum_{j=1}^{L} g_j(s, a, \theta)\lambda_j(\cdot|s, \theta)$, where*
- *for $j = 1, \dots, L$ function $g_j : S \times A \times \Theta \to [0, 1]$ is continuous with a for a given (s, θ), measurable (s, a) for given θ, increasing in (s, a, θ), supermodular in a for fixed (s, θ), and has increasing differences in $(a; s, \theta)$ and $g_j(0, a, \theta) = 0$ with $\sum_{j=1}^{L} g_j(\cdot) + g_0(\cdot) \equiv 1$,*
- *$(\forall s \in S, \theta \in \Theta, j = 1, \dots, L)\lambda_j(\cdot|s, \theta)$ is a Borel transition probability on S, with each $\lambda_j(\cdot|s, \theta)$ stochastically increasing with (s, θ),*
- *δ_0 is a probability measure concentrated at point 0.*

Notice, in both of these assumptions, we have added increasing differences between actions and states (as, for example, in [16]).

Before we state our main result in this section, we first introduce some new notation. For a stochastic game evaluated at parameter $\theta \in \Theta$, denote the least and greatest equilibrium values as v_θ^* and w_θ^*, respectively. Further, for each of these extremal values, denote the associated least and greatest MSNE pure strategies as ϕ_θ^* and ψ_θ^*, respectively. Then, our first monotone equilibrium comparative statics result is given in the next theorem.

Theorem 3.4 *(Monotone equilibrium comparative statics). Let Assumptions 4 and 5 be satisfied. Then, the extremal equilibrium values $v_\theta^*(s)$, $w_\theta^*(s)$ are increasing on $S \times \Theta$. In addition, the associated extremal pure strategy stationary Markov Nash equilibrium $\phi_\theta^*(s)$ and $\psi_\theta^*(s)$ are increasing on $S \times \Theta$.*

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17 That is, $\tilde{A}_i(s, \theta)$ is *ascending in Veinott's strong set order* if for any $(s, \theta) \leqslant (s', \theta')$, $a_i \in \tilde{A}_i(s, \theta)$ and $a_i' \in \tilde{A}_i(s', \theta') \Rightarrow a_i \wedge a_i' \in \tilde{A}_i(s, \theta)$ and $a_i \vee a_i' \in \tilde{A}_i(s', \theta')$.

18 That is, $\tilde{A}_i$ is *expanding* if $s_1 \leqslant s_2$ and $\theta_1 \leqslant \theta_2$ then $\tilde{A}_i(s_1, \theta_1) \subseteq \tilde{A}_i(s_2, \theta_2)$.

In the literature on *infinite horizon* stochastic games with strategic complementarities, we are not aware of any analog result to the above concerning monotone equilibrium comparative statics as in Theorem 3.4. In particular, because of the non-constructive approach to the equilibrium existence problem (that is typically taken in the literature), it is difficult to obtain such a monotone comparative statics without fixed point uniqueness. Therefore, one key innovation of our approach of the previous section is that for the special case of our games where SMNE are monotone Markov processes, we are able to construct a sequence of parameterized monotone operators whose fixed points are extremal equilibrium selections. As the method is constructive, this also allows us to compute directly the relevant monotone selections from the set of MSNE.

Finally, we state results on dynamics and invariant distributions started from s_0 and governed by a MSNE and transition Q. Before this, let us mention that by our assumptions delta Dirac concentrated at 0 is an absorbing state (and hence we can have a trivial invariant distribution). As a result, we do not aim to prove a general result on the existence of an invariant distribution; rather, we will characterize a set of all invariant distributions, and discuss conditions when this limiting distribution is not a singleton. Along these lines, let θ be given, and let s_t^f denote a stochastic process induced by Q and equilibrium strategy f (i.e., $s_0 = s^f$ is an initial value and for $t > 0$), with s_{t+1} has a conditional distribution $Q(\cdot|s_t, f(s_t))$. By $\succeq$ we denote the first order stochastic dominance order on the space of probability measures.

We then have the following theorem:

Theorem 3.5 *(Invariant distribution). Let Assumptions 1, 2 and 3 be satisfied.*

- *Then the sets of invariant distributions for processes $s_t^{\phi^*}$ and $s_t^{\psi^*}$ are chain complete (with both greatest and least elements) with respect to (first) stochastic order.*
- *Let $\bar{\eta}(\phi^*)$ be the greatest invariant distribution with respect to ϕ^* and $\bar{\eta}(\psi^*)$ the greatest invariant distribution with respect to ψ^*. If the initial state of $s_t^{\phi^*}$ or $s_t^{\psi^*}$ is a Dirac delta in $\bar{S}$, then $s_t^{\phi^*}$ converges weakly to $\bar{\eta}(\phi^*)$, and $s_t^{\psi^*}$ converges weakly to $\bar{\eta}(\psi^*)$, respectively.*[19]

We make a few remarks. First, the above result is stronger than that obtained in a related theorem in [16] (e.g., Theorem 5.2). That is, not only we do characterize the set of invariant distributions associated with extremal strategies (which he does not), but we also prove a weak convergence result per the greatest invariant selection. Second, it is worth mentioning if for almost all $s \in S$, we have $\sum_j g_j(s, \cdot) < 1$, we obtain a positive probability of reaching zero (an absorbing state) each period, and hence the only invariant distribution is delta Dirac at zero. Hence, to obtain a *nontrivial* invariant distribution, one has to assume $\sum_j g_j(s, \cdot) = 1$ for all s in *some subset* of a state space S with positive measure, e.g. interval $[S', \bar{S}] \subset S$ (see [28]).[20]

Second, Theorems 3.4 and 3.5 also imply results on monotone comparative dynamics (e.g., as defined by Huggett [27]) with respect to the parameter vector $\theta \in \Theta$ induced by extremal MSNE: ϕ^*, ψ^*. To see this, we define the greatest invariant distribution $\bar{\eta}_\theta(\phi_\theta^*)$ induced by $Q(\cdot|s, \phi_\theta^*, \theta)$, and greatest invariant distribution $\bar{\eta}_\theta(\psi_\theta^*)$ induced by $Q(\cdot|s, \psi_\theta^*, \theta)$, and consider the following corollary:

[19] That is their distributions converge weakly.

[20] It is also worth mentioning that much of the existing literature does not consider the question of characterizing the existence of invariant distributions.

Corollary 3.2. *Assume Assumptions 4, 5. Additionally let assumptions of Theorem 3.5 be satisfied for all $\theta \in \Theta$. Then $\bar{\eta}_{\theta_2}(\phi^*_{\theta_2}) \succeq \bar{\eta}_{\theta_1}(\phi^*_{\theta_1})$ as well as $\bar{\eta}_{\theta_2}(\psi^*_{\theta_2}) \succeq \bar{\eta}_{\theta_1}(\psi^*_{\theta_1})$ for any $\theta_2 \geqslant \theta_1$.*

We conclude with two remarks. First, the results above point to the importance of having constructive iterative methods for *both* strategies/values, as well as limiting distributions associated with extremal SMNE. That is, without such monotone iterations, we could not sharply characterization the *computation* of our monotone comparative statics results.

Second, we stress the fact that by weak continuity of operators used to establish invariant distributions, we can also obtain results that lead us to develop methods to *estimate* parameters θ using simulated moments methods (e.g., see [2], for discussion of how this is done, and why it is important).

4. Applications

There are many applications of our new results. For example, they can be used to study many examples such as dynamic (price or quantity) oligopolistic competition, stochastic growth models without commitment (and related problems of dynamic consistency), models with weak social interaction among agents, dynamic policy games, and interdependent security systems. In this section, we discuss three such applications of our results to the existence of Markov equilibrium in a dynamic oligopoly model, the analysis of credible government public policies as in [45], and the generalization of existing results per symmetric MSNE of symmetric stochastic games.

4.1. Price competition with durable goods

We begin with a price competition problem with durable goods. Consider an economy with n firms competing on customers buying durable goods, that are heterogeneous but substitutable to each other. Apart from price of a given good, and vector of competitors goods' prices, demand for any commodity depends on demand parameter s. Each period firms choose their prices, competing a la Bertrand with other's prices. Our aim is to analyze the Markov stationary Nash equilibrium of such economy.

Payoff of firm i, choosing price $a_i \in [0, \bar{a}]$ is

$$u_i(s, a_i, a_{-i}, \theta) = a_i D_i(a_i, a_{-i}, s) - C_i\big(D_i(a_i, a_{-i}, s), \theta\big),$$

where s is a (common) demand parameter, while θ is a cost function parameter. As within period game is Bertrand with heterogeneous, but substitutable products, naturally the preference Assumption 1 is satisfied if (a) demand D_i is increasing with a_{-i}, has increasing differences in (a_i, a_{-i}), and (b) the cost function C_i is increasing and convex. As $[0, \bar{a}]$ is single dimensional, u_i is a supermodular in a_i trivially.

Concerning the interpretation of the assumptions placed on Q in the context of this model: letting $s = 0$ be an absorbing state means that there is a probability that demand will vanish and companies will be driven out of the market. The other assumptions on transition probabilities are also satisfied if $Q(\cdot|s, a) = g_0(s, a)\delta_0(\cdot) + \sum_j g_j(s, a)\lambda_j(\cdot|s)$ and g_j, λ_j satisfy Assumption 2. We can interpret this assumption economically as follows: high prices a today result in high probability for positive demand in the future, as the customer trades-off between exchanging the old product with the new one, and keeping the old product and waiting for lower prices tomorrow. Supermodularity in prices implies that the impact of a price increase on positive demand parameter tomorrow is higher when the others set higher prices. Indeed, when the company increases its price today, it may lead to a positive demand in the future (if the others have also high prices). But, if the others firms set low prices today, then such impact is definitely lower, as some clients may want to purchase the competitors good today instead. Such assumptions guarantee that the stochastic (extensive form) game has the supermodular structure for extremal strategies, the feature that is uncommon for general extensive form games (see [20]). More specifically, if a strategy of a player is increased in some period $t + \tau$, it leads to a higher value of all players and by our mixing transition assumption increase period t extremal strategies.

The results of the paper (Theorem 3.1) prove existence of the greatest and the least Markov stationary Bertrand equilibrium and allow to compute these equilibria, by a simple iterative procedure. Our theorems extend, therefore, the results obtained in [16] to the non-monotone strategies, characterizing the monopolistic competition economy with substitutable durable goods and varying consumer preferences. Finally, our approximation procedure allows applied researchers to compute and estimate the stochastic properties of this model using the extremal invariant distributions (see Theorem 3.5). Finally, if one adds assumptions of Theorem 3.4 one obtains monotone comparative statics of the extremal equilibria and invariant distributions (see Corollary 3.2), the results absent in the related work.

Note, to analyze such an economy using the methods of [16], one needs to assume increasing differences between (a_i, s) and monotonicity in s. Such method allows hence to study the monotone equilibria only. The interpretation of such assumption means that high demand today imply high demand in the future. To justify this assumption, Curtat argues: that “high level of demand today is likely to result in a high level of demand tomorrow because one can assume that not all customers will be served today in the case of high demand”. Hence, in this paper, the existence of SMNE is obtained under weaker complementarity assumptions than the results in [16], e.g., in situations where monotonicity assumptions are not applicable, as customer rationing is not a part of this game description. Hence, methods developed in Section 3.2 are plausible.

4.2. Time-consistent public policy

We now consider a time-consistent policy game as defined by Stokey [45] and analyzed more recently by Lagunoff [29]. Consider a (stochastic) game between a large number of identical households and the government. We will study equilibria that treat each household identically. For any state $k \in S$ (capital level), households choose consumption c and investment i treating level of a government spending G as given. There are no security markets that household can share the risk for tomorrow capital level. The only way to consume tomorrow is to invest in the stochastic technology Q. The within period preferences for the households are given by $u(c)$ (i.e. household does not obtain utility from public spending G). The government raises revenue by levying flat tax $\tau \in [0, 1]$ on capital income, to finance its public spending $G \geqslant 0$. Each period the government budget is balanced and its within period preferences are given by: $u(c) + J(G)$. The consumption good production technology is given by constant return to scale function $f(k)$ with $f(0) = 0$. The transition technology between states is given by a probability distribution $Q(\cdot|i, k)$, where i denotes household investment. The timing of the game in each period is that the government and household choose their actions simultaneously. Observe, in this example a natural absorbing state is $k = 0$.

To specify each players optimization problem, first we assume households and the government take price R as given, with profit maximization implying $R = f'(k)$. Assume that u, J, f are increasing, concave and twice continuously differentiable and Q is given by Assumption 2 (with $L = 1$ to simplify notation). Each of the households then chooses investment i to solve:

$$\max_{i\in[0,(1-\tau)Rk]} u\big((1-\tau)Rk - i\big) + g(i)\beta \int_S v_H(s)\lambda(s|k).$$

By a standard argument, we see that the objective for the households is supermodular in i and has increasing differences in (i,t), where $t = 1-\tau$ (noting $-u''(\cdot) \geqslant 0$). Moreover, the objective is increasing in $t = 1-\tau$ by monotonicity assumptions on u.

The government is choosing t to solve:

$$\max_{t\in[0,1]} u(tRk - i) + J\big(Rk(1-t)\big) + g(i)\beta \int_S \big(v_H(s) + v_G(s)\big)\lambda(s|k).$$

That is, the government maximizes the household utility as well as the additional utility that it obtains from public spending J, and its continuation v_G. Again, objective is supermodular in $1-\tau$ and has increasing differences in $(t = 1-\tau, i)$ as $-u''(\cdot) \geqslant 0$. Moreover observe, although the objective is not increasing in i, along any Nash equilibrium of the auxiliary game, the government's objective is increasing in v_H by the envelope theorem. To see that, by $(i^*, t^*)(v_H)$ denote an extremal NE of the auxiliary game and observe that:

$$\begin{aligned}
&\frac{\partial}{\partial i}\left[u\big(t^*(v_H)Rk - i\big) + J\big(Rk\big(1-t^*(v_H)\big)\big) + g(i)\beta\int_S \big(v_H(s)+v_G(s)\big)\lambda(s|k)\right]_{i=i^*(v_H)}\\
&\quad= \left[-u'\big(t^*(v_H)Rk - i^*(v_H)\big) + g'\big(i^*(v_H)\big)\beta\int_S v_H(s)\lambda(s|k)\right]\\
&\qquad + g'\big(i^*(v_H)\big)\beta\int_S v_G(s)\lambda(s|k)\\
&\quad= g'\big(i^*(v_H)\big)\beta\int_S v_G(s)\lambda(s|k) \geqslant 0.
\end{aligned}$$

So, interestingly, although this model's general assumptions do not appear to satisfy the underlying sufficient conditions given in our paper, the same method developed in the paper can be extended easily to this case. That is, we are able to use our results to prove existence of MSNE, as well as compute the least and the greatest MSNE. Specifically, we can construct an operator on the space of values that would be monotone (as the within period game is supermodular and the Nash equilibrium of such game is monotone in v_H, v_G).

Some additional interesting points of departure from this above basic specification can also be worked out, including: (i) elastic labor supply choice, or more importantly (ii) adding security markets, investment/insurance firms possessing Q and proving existence of prices decentralizing optimal investment decision i^*. Still observe, however, that here we are able to offer weak assumptions for existence of a stationary credible policy, as well as offer a variety of tools allowing for its constrictive study and computation.

4.3. Symmetric equilibria in symmetric stochastic games

Finally, we consider a special case of our stochastic game, namely, one where all players have identical preferences $u := u_i$ and action sets $\tilde{A} := \tilde{A}_i \subset \mathbb{R}$. With slight abuse of notation, we denote payoff of a player choosing a_i, when others choose a_{-i} in state s by $u(s, a_i, a_{-i})$. Now observe that for such a special case we can obtain results of Theorem 3.1, Corollary 3.1 and others from Section 3.2 for *symmetric equilibria* dispensing Assumption 1 of increasing differences of u in (a_i, a_{-i}) and supermodularity of g in a in Assumption 2. Instead, to guarantee existence of the NE of the auxiliary game we need to add concavity of g in a, and concavity of u in a_i. Indeed, under such additional assumptions the auxiliary game G_v^s has the greatest and the least symmetric Nash equilibrium, both monotone in v by Corollary 2 of Milgrom and Roberts [35]. Hence, we can still construct two monotone operators $\bar{T}$, $\underline{T}$ and reconstruct the proofs of Theorem 3.1 and Corollary 3.1. Such modification is important, as it allows one to dispense with the restrictive assumption of (within period) strategic complementarities between players while allowing to obtain (between period) strategic complementarities (at least for selected extremal NE values), a necessary feature for our constructive arguments.

An immediate example of the importance of this generalization can be seen, when studying of symmetric SMNE in a stochastic version of a private provision of public good game. Let $u(c_i, Y)$ be a payoff from consumption of a private c_i and public good Y. Assume marginal utilities are decreasing, and both goods are complements. Endow consumer with income w to be distributed between c_i and private provision y_i. Let a public good be produced using technology $Y = F(\sum_i y_i, s)$, where F is increasing and concave in the first argument. Observe that the function $(y_i, y_{-i}) \to u(w - y_i, F(\sum_j y_j, s))$ does not have increasing differences, but has positive externalities due to free rider problem. Let s parameterize public good stock (i.e. a draw representing a stock from the previous period) or its productivity, while Q represent a process allowing to reduce a future probability of a zero output/productivity, by higher provisions $(y_1, \ldots, y_n)$ today. By Theorem 3.1 and Corollary 3.1 we can prove existence and approximate the greatest and least symmetric MSNE of such a game.

Finally, using this generalization, we can reconsider symmetric MSNE of a Bertrand competition with durable good example (see Section 4.1), and relax increasing differences assumption of demand D_i with (p_i, p_{-i}) and supermodularity of g.

5. Proofs

We first begin by stating a few lemmata that prove useful in verifying the existence of SMNE in our game, as well as characterizing monotone iterative procedures for constructing least and greatest SMNE (relative to pointwise partial orders on $\mathcal{B}^n(S)$). More specifically, these lemmas concern the structure of Nash equilibria (and their associated corresponding equilibrium payoffs) in our auxiliary game G_v^s.

Lemma 5.1 *(Monotone Nash equilibria in G_v^s). Under Assumptions 1 and 2, for every $s \in S$ and value $v \in \mathcal{B}^n(S)$, the game G_v^s has the maximal Nash equilibrium $\bar{a}(v, s)$, and minimal Nash equilibrium $\underline{a}(v, s)$. Moreover, both equilibria are increasing in v.*

Proof. Without loss of generality fix $s > 0$. Define the auxiliary one shot game, say $\Delta(\tau)$, with an action space A, and payoff function for player i given as

$$H_i(a, \tau) := (1 - \beta_i)u_i(s, a_i, a_{-i}) + \beta_i \sum_{j=1}^{L} \tau_{i,j} g_j(s, a_i, a_{-i}),$$

where $\tau := [\tau_{i,j}]_{i=1,\ldots,n, j=1,\ldots,L} \in \mathcal{T} := \mathbb{R}^{n \times L}$ is endowed with the natural pointwise order. As supermodularity of a function on a sublattice of a directed product of lattices implies increasing differences (see [49, Theorem 2.6.1]), for each $\tau \in \mathcal{T}$, the game $\Delta(\tau)$ is supermodular, and satisfies all assumptions of Theorem 5 in [34]. Hence, there exists a complete lattice of Nash equilibria, with the greatest Nash equilibrium given by $\overline{\text{NE}}\Delta(\tau)$, and the least Nash equilibrium given by $\underline{\text{NE}}\Delta(\tau)$. Moreover, for arbitrary i, the payoff function $H_i(a,\tau)$ has increasing differences in a_i and τ; hence, $\Delta(\tau)$ also satisfies conditions of Theorem 6 in [34]. As a result, both $\overline{\text{NE}}\Delta(\tau)$ and $\underline{\text{NE}}\Delta(\tau)$ are increasing in τ.

Step 2: For each $s \in S$, the game G_v^s is a special case of $\Delta(\tau)$ where $\tau_{i,j} = \int_S v_i(s')\lambda_j(ds'|s)$. Therefore, by the previous step, least and greatest Nash equilibrium $\underline{a}(v,s)$ and $\bar{a}(v,s)$ are increasing in v, for each $s \in S$ □

In our next lemma, we show that for each extremal Nash equilibrium (for state s and continuation v), we can associate an equilibrium payoff that preserves monotonicity in v. To do this, we first compute the values of greatest (resp., least) best responses given a continuation values v and state s as follows:

$$\bar{\Pi}_i^*(v,s) := \Pi_i\big(v_i, s, \bar{a}_i(v,s), \bar{a}_{-i}(v,s)\big),$$

and similarly

$$\underline{\Pi}_i^*(v,s) := \Pi_i\big(v_i, s, \underline{a}_i(v,s), \underline{a}_{-i}(v,s)\big).$$

We now have the following lemma:

Lemma 5.2 *(Monotone values in G_v^s). Under Assumptions 1 and 2 we have: $\bar{\Pi}_i^*(v,s)$ and $\underline{\Pi}_i^*(v,s)$ are monotone in v.*

Proof. Function Π_i is increasing with a_{-i} and v_i. For $v_2 \geqslant v_1$ by Lemma 5.1, we have $\underline{a}(v_2,s) \geqslant \underline{a}(v_1,s)$. Hence,

$$\begin{aligned}\underline{\Pi}_i^*\big(v^2,s\big) &= \max_{a_i \in \tilde{A}_i(s)} \Pi_i\big(v_i^2, s, a_i, \underline{a}_{-i}\big(v^2,s\big)\big) \geqslant \max_{a_i \in \tilde{A}_i(s)} \Pi_i\big(v_i^1, s, a_i, \underline{a}_{-i}\big(v^2,s\big)\big) \\ &\geqslant \max_{a_i \in \tilde{A}_i(s)} \Pi_i\big(v_i^1, s, a_i, \underline{a}_{-i}\big(v^1,s\big)\big) = \underline{\Pi}_i^*\big(v^1,s\big).\end{aligned}$$

A similar argument proves the monotonicity of $\bar{\Pi}_i^*(v,s)$. □

To show that $\bar{T}(\cdot)(s) = \bar{\Pi}(\cdot,s)$ and $\underline{T}(\cdot)(s) = \underline{\Pi}(\cdot,s)$ are well-defined transformations of $\mathcal{B}^n(S)$ we use standard measurable selection arguments.

Lemma 5.3 *(Measurable equilibria and values of G_v^s). Under Assumptions 1 and 2 we have:*

- *$\bar{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$ and $\underline{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$,*
- *functions $s \to \bar{a}(v,s)$ and $s \to \underline{a}(v,s)$ are measurable for any $v \in \mathcal{B}^n(S)$.*

Proof. For $v \in \mathcal{B}^n(S)$ and $s \in S$, define the function $F_v : A \times S \to \mathbb{R}$ as follows:

$$F_v(a,s) = \sum_{i=1}^n \Pi_i(v_i, s, a) - \sum_{i=1}^n \max_{z_i \in \tilde{A}_i(s)} \Pi_i(v_i, s, z_i, a_{-i}).$$

Observe $F_v(a,s) \leqslant 0$. Consider the problem:

$$\max_{a\in\times_i^n \tilde{A}_i(s)} F_v(a,s).$$

By Assumption 1 and 2, the objective F_v is a Carathéodory function, and the (joint) feasible correspondence $\tilde{A}(s) = \times_i \tilde{A}_i(s)$ is weakly-measurable. By a standard measurable maximum theorem (e.g. Theorem 18.19 in [3]), the correspondence $N_v : S \twoheadrightarrow \times_i A_i(s)$ defined as:

$$N_v(s) := \arg\max_{a\in\tilde{A}_i(s)} F_v(a,s),$$

is measurable with nonempty compact values. Further, observe that $N_v(s)$, by definition, is a set of all Nash equilibria for the game G_v^s. Therefore, to finish the proof of our first assertion, for some player i, consider a problem $\max_{a\in N_v(s)} \Pi_i(v_i,s,a)$. Again, by the measurable maximum theorem, the value function $\bar{\Pi}_i^*(v,s)$ is measurable. A similar argument shows each $\underline{\Pi}_i^*(v,s)$ is measurable. Therefore the product operators are also measurable, giving: $\bar{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$ and $\underline{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$.

To show the second assertion of the theorem, for some player i, again consider a problem of $\max_{a\in N_v(s)} a_i^j$ for some $j \in \{1,2,\ldots,k_i\}$. Again, appealing to the measurable maximum theorem and Theorem 4.1 in [26], the product of (maximizing) selections $\bar{a}(v,s)$ (resp., $\underline{a}(v,s)$) is measurable with s. □

Proof of Lemma 3.1. Proof of 1: Clearly $\phi^1 \leqslant \phi^2$ and $v^1 \leqslant v^2$. Suppose $\phi^j \leqslant \phi^{j+1}$ and $v^j \leqslant v^{j+1}$. By the definition of the sequence $\{v^j\}$ and Lemma 5.2, we have $v^{j+1} \leqslant v^{j+2}$. Then, by Lemma 5.1, definition of $\{\phi^j\}$, and the induction hypotheses, we obtain $\phi^{j+1}(s) = \underline{a}(v^{j+1},s) \leqslant \underline{a}(v^{j+2},s) = \phi^{j+2}(s)$. Similarly, we obtain monotonicity of ψ^j and w^j.

Proof of 2: Clearly, the thesis is satisfied for $j=1$. By induction, suppose that the thesis is satisfied for some j. Since $v^j \leqslant w^j$, by Lemma 5.2, we obtain

$$v^{j+1}(s) = \underline{\Pi}^*\big(v^j,s\big) \leqslant \underline{\Pi}^*\big(w^j,s\big) \leqslant \bar{\Pi}^*\big(w^j,s\big) = w^{j+1}(s).$$

Then, by Lemma 5.1, we obtain

$$\begin{aligned}\phi^{j+1}(s) &= \underline{a}\big(v^{j+1},s\big)\\ &\leqslant \underline{a}\big(w^{j+1},s\big) \quad \text{and hence}\\ &\leqslant \bar{a}\big(w^{j+1},s\big) = \psi^{j+2}(s).\end{aligned}$$

Proof of 3–4: It is clear since for each $s\in S$, the sequences of values v^j, w^j and associated pure strategies ϕ^j and ψ^j are bounded. Further, by previous step, they are monotone.

Proof of 5: By definition of v^j and ϕ^j, we obtain

$$\begin{aligned}v_i^{j+1}(s) &= (1-\beta_i)u_i\big(s,\phi^j(s)\big) + \beta_i \sum_{j=1}^{L} g_j\big(s,\phi^j(s)\big) \int_S v_i^j\big(s'\big)\lambda_j\big(ds'\big|s\big)\\ &\geqslant (1-\beta_i)u_i\big(s,a_i,\phi_{-i}^j(s)\big) + \beta_i \sum_{j=1}^{L} g_j\big(s,a_i,\phi_{-i}^j(s)\big) \int_S v_i^j\big(s'\big)\lambda_j\big(ds'\big|s\big),\end{aligned}$$

for arbitrary $a_i \in \tilde{A}_i(s)$. By the continuity of u_i and g and the Lebesgue Dominance Theorem, if we take a limit $j\to\infty$, we obtain

$$v_i^*(s) = (1-\beta_i)u_i\big(s,\phi^*(s)\big) + \beta_i \sum_{j=1}^{L} g_j\big(s,\phi^*(s)\big) \int_S v_i^*\big(s'\big)\lambda_j\big(ds'\big|s\big)$$
$$\geqslant (1-\beta_i)u_i\big(s,a_i,\phi_{-i}^*(s)\big) + \beta_i \sum_{j=1}^{L} g_j\big(s,a_i,\phi_{-i}^*(s)\big) \int_S v_i^*\big(s'\big)\lambda_j\big(ds'\big|s\big),$$

which, by Lemma 5.3, implies that ϕ^* is a pure stationary (measurable) Nash equilibrium, and v^* is its associated (measurable) equilibrium payoff. Analogously, we have ψ^* a pure strategy (measurable) Nash equilibrium, and w^* its associated (measurable) equilibrium payoff. □

Proof of Theorem 3.1. The first and third point results from Lemma 3.1. To prove the second one we proceed in steps. Step 1. We prove the desired inequality for equilibria payoffs. Since $0 \leqslant \mu^* \leqslant \bar{u}$, by Lemma 5.2 and definition of v^t and w^t, we obtain

$$v_1 \leqslant \mu^* \leqslant w_1.$$

By induction, let $v_t \leqslant \mu^* \leqslant w_t$. Again, from Lemma 5.2 we have:

$$v_{t+1} = \underline{\Pi}^*(v_t, s) \leqslant \underline{\Pi}^*\big(\mu^*, s\big) \leqslant \mu^*(s) \leqslant \bar{\Pi}^*\big(\mu^*, s\big) \leqslant \bar{\Pi}^*(w_t, s) = w_{t+1}.$$

Taking a limit with t we obtain desired inequality for equilibria payoffs.

Step 2: By previous step and Lemma 5.1, we obtain:

$$\phi^*(s) = \underline{a}\big(v^*, s\big) \leqslant \underline{a}\big(\mu^*, s\big) \leqslant \gamma^*(s) \leqslant \bar{a}\big(\mu^*, s\big) \leqslant \bar{a}\big(w^*, s\big) = \psi^*(s). \qquad \square$$

For fixed continuation value v let:

$$M_{i,j,l} := \sup_{s\in S, a\in A(s)} \frac{\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_l}}{-\sum_{\tilde{i}=1}^{n}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}}},$$
$$M := \max\{M_{i,j,l} : i = 1,\dots,n, \text{ and } j = 1,\dots,k_i, \text{ and } l = 1,\dots,k\}.$$

By Assumption 3, the constant M is a strictly positive real number.

Lemma 5.4. *Let Assumptions 1, 2, 3 be satisfied and constraint functions $\tilde{a}_i \in CM^{k_i}$. Fix $v \in \mathcal{B}_n(S)$, and assume it is Lipschitz continuous. Consider an auxiliary game G_v^s. Then, there is a unique Nash equilibrium in this game $a^*(v,s)$ and belongs to $CM^{\sum_i k_i}$.*

Proof. Let $s > 0$ and let $v \in \mathcal{B}^n(S)$ be Lipschitz continuous function. To simplify we drop v from our notation. Let $x^1(s) = \tilde{a}(s)$ and $x_i^{t+1}(s) := \arg\max_{a_i \in \tilde{A}_i(s)} \Pi_i(s, a_i, x_{-i}^t(s))$ for $n \geqslant 1$. This is well defined by strict concavity of Π_i in a_i. Clearly, x^1 is nondecreasing and Lipschitz continuous with a constant less that M. By induction, assume that this thesis holds for $t \in \mathbb{N}$. Note that $(s, a_i) \to \Pi_i(s, a_i, x_{-i}^t(s))$ has increasing differences. Indeed, if we take $s_1 \leqslant s_2$ and $y_1 \leqslant y_2$, then $x_{-i}^t(s_1) \leqslant x_{-i}^t(s_2)$ and

$$\Pi_i\big(s_1, y_2, x_{-i}^t(s_1)\big) - \Pi_i\big(s_1, y_1, x_{-i}^t(s_1)\big)$$
$$\leqslant \Pi_i\big(s_1, y_2, x_{-i}^t(s_2)\big) - \Pi_i\big(s_1, y_1, x_{-i}^t(s_2)\big)$$
$$\leqslant \Pi_i\big(s_2, y_2, x_{-i}^t(s_2)\big) - \Pi_i\big(s_2, y_1, x_{-i}^t(s_2)\big).$$

Therefore, since $\tilde{A}_i(\cdot)$ is ascending in the Veinott strong set order, by Theorem 6.1 in [47] we obtain that $x_i^{t+1}(\cdot)$ is isotone. We next show that $x_i^{t+1}(\cdot)$ is Lipschitz continuous with a constant M. To do this we check hypotheses of Theorem 2.4(ii) in [16]. Define $\varphi(s) = s_1 + \cdots + s_k$. Define $\mathbf{1}_i := (1, 1, \dots, 1) \in \mathbb{R}^{k_i}$. We show that the function $(s, y) \to \Pi^*(s, y) := \Pi_i(s, M\varphi(s)\mathbf{1}_i - y, x_{-i}^t(s))$ has increasing differences. Note that $M\varphi(s) - \tilde{a}_i(s) \leqslant y \leqslant M\varphi(s)$. We show that the collection of the sets $Y(s) := [M\varphi(s) - \tilde{a}_i(s), M\varphi(s)]$ is ascending in the Veinott strong set order. Let $s_1 \geqslant s_2$ in product order. Then,

$$M\varphi(s_1) - \tilde{a}_i^j(s_1) - \big(M\varphi(s_2) - \tilde{a}_i^j(s_2)\big) = M\|s_1 - s_2\|_1 - \big|\tilde{a}_i^j(s_1) - \tilde{a}_i^j(s_2)\big| \geqslant 0.$$

This, therefore, implies that lower bound of $Y(s)$ is increasing with s. Clearly upper bound of $Y(s)$ is increasing as well. Hence $Y(s)$ is ascending in the Veinott's strong set order.

Note that since for all $s_l \to x^t(s)$ is monotone and continuous, hence must be differentiable almost everywhere [43]. By M-Lipschitz property of x^t we conclude that each partial derivative is bounded by M. Hence we have for all $l = 1, \dots, k$, $i = 1, \dots, n$ and $j = 1, \dots, k_i$:

$$\frac{\partial \Pi_i^*}{\partial y_i^j} = -\frac{\partial \Pi_i}{\partial a_i^j}.$$

Next we have (for fixed s_{-k}):

$$\begin{aligned}
\frac{\partial^2 \Pi_i^*}{\partial y_i^j \partial s_k} &= -\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_k} - M \sum_{\alpha=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a_i^j a_i^\alpha} \frac{\partial \varphi}{\partial s_k} - \sum_{\tilde{i} \neq i} \sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}} \frac{\partial x_{\tilde{i},\tilde{j}}^*}{\partial s_k} \\
&\geqslant -\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_k} - M \sum_{\alpha=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a_i^j a_i^\alpha} - \sum_{\tilde{i} \neq i} \sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}} M \\
&= -\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_k} - \sum_{\tilde{i}=1} \sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}} M \\
&= \Bigg(-\sum_{\tilde{i}=1} \sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}}\Bigg)\Bigg(M - \frac{\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_k}}{-\sum_{\tilde{i}=1} \sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}}}\Bigg) \geqslant 0
\end{aligned}$$

almost everywhere. Since $\frac{\partial \Pi_i^*}{\partial a_i^j}$ is continuous, by Theorem 6.2 in [47] the solution of the optimization problem $y \to \Pi^*(v, s, y)$ (say $y^*(s, v)$) is isotone. From definition of $y^*(s, v)$ and x^{t+1} if $s_1 \leqslant s_2$ we have

$$0 \leqslant x_{i,j}^{t+1}(s_1) - x_{i,j}^{t+1}(s_2) \leqslant M\big(\varphi(s_1) - \varphi(s_2)\big) = M\|s_1 - s_2\|_1.$$

Analogously we prove appropriate inequality whenever $s_1 \geqslant s_2$. If s_1 and s_2 are incomparable then

$$x_{i,j}^{t+1}(s_1) - x_{i,j}^{t+1}(s_2) \leqslant x_{i,j}^{t+1}(s_1 \vee s_2) - x_{i,j}^{t+1}(s_1 \wedge s_2) \leqslant M\|s_1 - s_2\|_1$$

since $\|s_1 - s_2\|_1 = \|s_1 \vee s_2 - s_1 \wedge s_2\|_1$. Similarly we prove that:

$$x_{i,j}^{t+1}(s_1) - x_{i,j}^{t+1}(s_2) \geqslant -M\|s_1 - s_2\|_1.$$

But this implies that x^{t+1} is also M-Lipschitz continuous, which implies that each x^t is M-Lipschitz continuous. Since Π_i has increasing differences in (a_i, a_{-i}) hence by Theorem 6.2 in [47] we know that the operator $x \to \arg\max_{y_i \in \tilde{A}_i(s)} \Pi_i(s, y_i, x_{-i})$ is increasing. Therefore $x^t(s)$ must be decreasing in t. This implies that there exists $a^* = \lim_{n\to\infty} x^t$ which is isotone and M-Lipschitz continuous. Uniqueness of Nash Equilibria follows from Assumption 3 and [22], hence $a^* = a^*(s, v)$ for $s > 0$.

Finally $\Pi(0, a) = 0$ for all $a \in A$ hence we can define $a^*(0) := \lim_{s\to 0^+} a^*(s)$ and obtain a unique Nash equilibrium $a^*(s, v)$ that is isotone and M-Lipschitz continuous in s. □

Proof of Theorem 3.2. To simplify notation, let $L = 1$ and hence $g(s, a) := g_1(s, a)$ and $\eta(f)(s) := \eta_1^f(s)$. Let $v \in \mathcal{B}^n(S)$ be Lipschitz continuous. Under Assumptions 3, $a^*(s, v)$ is a well defined (for $s > 0$) as auxiliary game satisfies conditions of [22]. Let $\pi_i(v_i, s, a) = (1 - \beta_i)u_i(s, a) + \beta_i\eta_i(v_i)g(s, a)$, and observe that $\pi_i(v_i, s, \cdot)$ has also strict diagonal property, and obviously has cardinal complementarities. Here, note that

$$\mathcal{L}_{i,j}\big(\pi_i(v_i, s, \cdot)\big) = (1 - \beta_i)\mathcal{L}_{i,j}\big(u_i(s, \cdot)\big) + \beta_i\eta_i(v_i)(s)\mathcal{L}_{i,j}\big(g(s, \cdot)\big) < 0.$$

Note further that applying Royden [43] result and continuity of the left side of expression below we have

$$\frac{\frac{\partial^2 \pi_i(v,s,\cdot)}{\partial a_i^j \partial s_l}}{-\sum_{\tilde{i}=1}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \pi_i(v,s,\cdot)}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}}} = \frac{(1-\beta_i)\frac{\partial u_i}{\partial a_i^j \partial s_l} + \beta_i \frac{\partial \eta(v)(s)}{\partial s_l}\frac{\partial g(s,a)}{\partial a_i^j} + \beta_i \eta(v)(s)\frac{\partial^2 g(s,a)}{\partial a_i^j \partial s_l}}{-(1-\beta_i)\mathcal{L}_{i,j}(u_i) - \beta_i\eta(v)(s)\mathcal{L}_{i,j}(g)}$$
$$\leqslant \frac{(1-\beta_i)U^2_{i,j,l} + \beta_i\bar{\eta}G^1_{i,j} + \beta_i\bar{u}G^2_{i,j,l}}{-(1-\beta_i)\mathcal{L}_{i,j}(u_i)} \leqslant M_0.$$

By Lemma 5.4, we know that $a^*(\cdot, v) \in CM_0^{\sum_i k_i}$. The following argument shows that Tv is Lipschitz continuous.

$$\begin{aligned}
\big|T_i v(s_1) - T_i v(s_2)\big| &\leqslant (1-\beta_i)\big|u_i\big(s_1, a^*(s_1, v)\big) - u_i\big(s_2, a^*(s_2, v)\big)\big| \\
&\quad + \beta_i\big|\eta(v_i)(s_1) - \eta(v_i)(s_2)\big|g\big(s_1, a^*(s_1, v)\big) \\
&\quad + \beta_i\eta(v_i)(s_2)\big|g\big(s_1, a^*(s_1, v)\big) - g\big(s_2, a^*(s_2, v)\big)\big| \\
&\leqslant \big(U_1 + M_0U_2 + \bar{\eta} + (G_1 + G_2M_0)\bar{u}\big)\|s_1 - s_2\|_1 \\
&= M_1\|s_1 - s_2\|_1,
\end{aligned}$$

where

$$U_1 := \sum_{l=1}^{h} \sup_{s\in S, a\in\tilde{A}(s)} \frac{\partial u_i}{\partial s_l}(s, a), \qquad U_2 := \sum_{i=1}^{m}\sum_{j=1}^{k_i} \sup_{s\in S, a\in\tilde{A}(s)} \frac{\partial u_i}{\partial a_i^j}(s, a),$$
$$G_1 := \sum_{l=1}^{h} \sup_{s\in S, a\in\tilde{A}(s)} \frac{\partial g}{\partial s_l}(s, a), \qquad G_2 := \sum_{i=1}^{m}\sum_{j=1}^{k_i} \sup_{s\in S, a\in\tilde{A}(s)} \frac{\partial g}{\partial a_i^j}(s, a),$$

and $M_1 := U_1 + M_0U_2 + \bar{\eta} + (G_1 + G_2M_0)\bar{u}$. Hence image of operator T is a subset of CM_1^n. Therefore the thesis is proven. □

Proof of Theorem 3.3. Let M be Lipschitz constant for equilibria values from Theorem 3.2. On CM^n define $T(v)(s) = \Pi^*(v, s)$. By a standard argument (e.g. [16]) $T : CM^n \to CM^n$ is continuous. By our lemma 5.2 it is also increasing on CM^n. By Tarski [46] theorem, it therefore has a nonempty complete lattice of fixed points, say $FP(T)$. Further, for each fixed point $v^*(\cdot) \in FP(T)$, there is a corresponding unique stationary Markov Nash equilibrium $a^*(v^*, \cdot)$. $\square$

Lemma 5.5. *Let X be a lattice, Y be a poset. Assume* (i) $F : X \times Y \to \mathbb{R}$ *and* $G : X \times Y \to \mathbb{R}$ *have increasing differences,* (ii) *that* $\forall y \in Y$, $G(\cdot, y)$ *and* $\gamma : Y \to \mathbb{R}_+$ *are increasing functions. Then, function H defined by $H(x, y) = F(x, y) + \gamma(y)G(x, y)$ has increasing differences.*

Proof. Under the hypotheses of the lemma, it suffices to show that $\gamma(y)G(x, y)$ has increasing differences (as increasing differences is a cardinal property and closed under addition). Let $y_1 > y_2$, $x_1 > x_2$ and $(x_i, y_i) \in X \times Y$. By the hypothesis of increasing differences of G, and monotonicity of γ and $G(\cdot, y)$, we have the following inequality

$$\gamma(y_1)\big(G(x_1, y_1) - G(x_2, y_1)\big) \geqslant \gamma(y_2)\big(G(x_1, y_2) - G(x_2, y_2)\big).$$

Therefore,

$$\gamma(y_1)G(x_1, y_1) + \gamma(y_2)G(x_2, y_2) \geqslant \gamma(y_1)G(x_2, y_1) + \gamma(y_2)G(x_1, y_2). \qquad \square$$

Proof of Theorem 3.4. *Step* 1: Let v_θ be a function $(s, \theta) \to v_\theta(s)$ that is increasing. By Assumptions 4, 5 and Lemma 5.5, the payoff function $\Pi_i(v_\theta, s, a, \theta)$ has increasing differences in (a_i, θ). Further, Π_i clearly also has increasing differences in (a_i, a_{-i}). As $\tilde{A}_i$ is ascending in Veinott's strong set order, by Theorem 6 in [34], the greatest and the least Nash equilibrium in the supermodular game $G^s_{v,\theta}$ are increasing selections. Further, by the same argument as in Lemma 5.2, as $\tilde{A}_i$ is also ascending under set inclusion by assumption, we obtain monotonicity of corresponding equilibria payoff.

Step 2: Note, for each θ, the parameterized stochastic game satisfies conditions of Theorem 3.1. Further, noting the initial values of the sequence of $w^t_\theta(s)$ and $v^t_\theta(s)$ (constructed in Theorem 3.1) do not depend on θ and are isotone in s, by the previous step, each iteration of both sequences of values is increasing with respect to (s, θ). Also, each of the iterations of $\phi^t_\theta(s)$ and $\psi^t_\theta(s)$ are also increasing in (s, θ). Therefore, as the pointwise partial order is closed, the limits of these sequences preserve this partial ordering, and the limits are increasing with respect to (s, θ). $\square$

For each equilibrium strategy f, define the operator

$$T^o_f(\eta)(A) = \int_S Q\big(A \big| s, f(s)\big)\eta(ds). \tag{1}$$

where η^* is said to be invariant with respect to f if and only if it is a fixed point of T^o_f.

Proof of Theorem 3.5. By Theorem 3.4 both ϕ^* and ψ^* are increasing functions. Let v be increasing function. By Assumption 2

$$\int_S v(s)T^o_{\phi^*}(\eta)(ds) = \sum_{j=1}^{L} g_j\big(s, \phi^*(s)\big) \int_S v(s')\lambda_j\big(ds' \big| s\big)\eta(ds).$$

Since by assumption, for each $s \in S$, the function under integral is increasing, the right-side increases pointwise whenever η stochastically increases. Moreover, as the family of probability measures on a compact state space S ordered by $\succeq$ (first order stochastic dominance) is chain complete (as it is a compact ordered topological space, e.g., [5, Lemma 3.1 or Corollary 3.2]). Hence, $T^o_{\phi^*}$ satisfies conditions of [32] theorem (Theorem 9), and we conclude that the set of invariant distributions is a chain complete with greatest and least invariant distributions (see also [5, Theorem 3.3]). By a similar construction, the same is true for the operator $T^o_{\psi^*}$.

To show the second assertion, we first prove that $T^o_{\phi^*}(\cdot)(A)$ is weakly continuous (i.e. if $\eta_t \to \eta$ weakly then $T^o_{\phi^*}(\eta_t) \to T^o_{\phi^*}(\eta)$ weakly). Let $\eta_t \to \eta$ weakly, and v be continuous. By Feller property of $\lambda_j(\cdot|s)$, we have $s \to \int_S v(s')\lambda_j(s'|s)$ continuous. Therefore,

$$\int_S \int_S v(s')\lambda_j(ds'|s)\eta_t(ds) \to \int_S \int_S v(s')\lambda_j(s'|s)\eta(ds).$$

This, in turn, implies

$$T^o_{\phi^*}(\eta_t) \to T^o_{\phi^*}(\eta)$$

weakly. Let $\eta_t^{\phi^*}$ be a distribution of $s_t^{\phi^*}$ and $\eta_1^{\phi^*} = \delta_{\bar{S}}$. By the previous step, η_t is stochastically decreasing. It is, therefore, weakly convergent to some η^*. By continuity of T^o, we have $\eta^* = T^o_{\phi^*}(\eta^*)$. By definition of $\bar{\eta}(\phi^*)$, we immediately obtain $\bar{\eta}(\phi^*) \preceq \eta^*$. By the stochastic monotonicity of $T^o_{\phi^*}(\cdot)$, we can recursively obtain that $\delta_{\bar{S}} \succeq \eta_t^{\phi^*} \succeq \bar{\eta}(\phi^*)$, and hence $\eta^* \succeq \bar{\eta}(\phi^*)$. As a result, we conclude $\eta^* = \bar{\eta}(\phi^*)$. Similarly, we show convergence of the sequence of distributions $s_t^{\psi^*}$. □

Proof of Corollary 3.2. By Theorem 3.5, there exists greatest fixed points for $T^{o,\theta_2}_{\phi^*}$ and $T^{o,\theta_1}_{\phi^*}$. Also, $T^{o,\theta}_{\phi^*}$ is weakly continuous. Further, $\theta \to T^{o,\theta}_{\phi^*}$ is an increasing map under first stochastic dominance on a chain complete poset of probability measures on the compact state set.

Consider a sequence of iterations from a $\delta_{\bar{S}}$ generated on $T^{o,\theta}_{\phi^*}$ (the operator defined in (1) but associated with $Q(\cdot|s,a,\theta)$). Observe, by the Tarski–Kantorovich theorem [19, Theorem 4.2], we have

$$\sup_t T^{t,o,\theta_2}_{\phi^*} = \bar{\eta}_{\theta_2} \quad \text{and} \quad \sup_t T^{t,o,\theta_2}_{\phi^*} = \bar{\eta}_{\theta_2}.$$

As for any t, we also have $T^{t,o,\theta_2}_{\phi^*} \succeq T^{t,o,\theta_1}_{\phi^*}$. Therefore, by weak continuity (and the fact that $\succeq$ is a closed order), we obtain:

$$\bar{\eta}_{\theta_2} = \sup_t T^{o,\theta_2}_{\phi^*} \succeq \sup_t T^{t,o,\theta_1}_{\phi^*} = \bar{\eta}_{\theta_1}.$$

Similarly we proceed for ψ^*. □

Remark 1. Since norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent on $\mathbb{R}^h$ and for all $s \in S$, $\|s\|_1 \leqslant h\|s\|$, hence we have that v^*, w^*, ϕ^* and ψ^* each Lipschitz continuous when we endow S with the maximum norm.

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