

Stationary Markov perfect equilibria in risk sensitive stochastic overlapping generations models [☆]

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Received 11 April 2013; final version received 29 November 2013; accepted 26 January 2014

Available online 31 January 2014

Abstract

In this paper, we study intergenerational stochastic games that can be viewed as a special class of overlapping generations models under uncertainty. Making use of the theorem of Dvoretzky, Wald and Wolfowitz [27] from the statistical decision theory, we obtain new results on stationary Markov perfect equilibria for the aforementioned games, with a general state space, satisfying rather mild continuity and compactness conditions. A novel feature of our approach is the fact that we consider risk averse generations in the sense that they aggregate partial utilities using an exponential function. As a byproduct, we also provide a new existence theorem for intergenerational stochastic game within the standard framework where the aggregator is linear. Our assumptions imposed on the transition probability and utility functions allow to embrace a pretty large class of intergenerational stochastic games analysed recently in macroeconomics. Finally, we formulate a set of assumptions under which the stochastic process induced by the stationary Markov perfect equilibrium possesses an invariant distribution.

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JEL classification: C62; C73; D91; O40

Keywords: Overlapping generations model; Intergenerational stochastic game; Risk sensitive optimisation; Stationary Markov perfect equilibrium

[☆] This work is partially supported by the National Science Centre: Grant DEC-2011/03/B/ST1/00325.

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1. Introduction

Intertemporal decision making that utilises standard optimisation criteria such as the total or discounted expected return over the time horizon may be quite insufficient to characterise the problem from the point of view of the decision maker. This is because these measures do not reflect the variability-risk features of the model. In order to circumvent the difficulty Howard and Matheson [47] studied a Markovian controlled model, in which the decision maker is equipped with a constant absolute risk coefficient. This assumption, in turn, implies that such a decision maker grades his/her utility via the expectations of the exponential function of future random outcomes. Moreover, the Taylor expansion of the exponential function allows to observe that such a decision maker does not only take into account the expectation of random returns received in the future, but also he/she values (with different weights) all their higher moments. This fact resulted in a rapid development of risk sensitive control theory in diverse research areas. In particular, the stimulating ideas have found a number of applications in macroeconomic models, see Tallarini [72], Hansen and Sargent [39], Anderson et al. [5] and references cited therein. The reader interested in further virtues and properties of this criterion is referred, e.g., to the comprehensive monograph of Whittle [75] and Föllmer and Schied [30].

A key feature of the aforementioned works is that they admit a (normative) representative decision maker. In many real life situations, however, the assumption of a representative decision maker is not acceptable. For example, the arrival of new decision makers in the economy is not only realistic but also yields a range of fresh economic interactions. These interactions are neatly captured in various overlapping generations (OLG) models. OLG models are most often used in macroeconomics, but they can also be useful in microeconomics (see, for example, Ljungquist and Sargent [54], Bewley [12], Acemoglu [1] or Geanakoplos [33], for general accounts). The baseline OLG model and its various deterministic variants gave rise to the extensive study of OLG models within a stochastic framework, see Peled [67], Duffie et al. [24] and references cited therein are good examples of works that deal with this issue. Specifically, Duffie et al. [24], making use of certain ideas from the theory of Markov processes on Borel state space and stochastic games, proved a general theorem on the existence of stationary Markov equilibrium that induces an ergodic process in an OLG model. Similar results on the existence of Markov perfect equilibrium for different OLG models were given in Gottardi [35] and Harris and Laibson [40], Krusell and Smith [51]. It is worthy to observe that although the latter two works are formulated in terms of “hyperbolic consumers” rather than generations, their results can be easily expressed for a certain type of an OLG model. A common feature of all aforementioned papers on stochastic OLG models is that they use additive aggregation of partial utilities concerning actions of following generations.

A major contribution of this paper is an application of principle ideas from risk sensitive control theory to certain classes of OLG models. To the best of our knowledge, such a merging of the two theories has not been encountered so far in the literature.

In this paper, we are concerned with a stochastic intergenerational bequest game inspired by the seminal work of Phelps and Pollak [68]. In their model, it is assumed that each generation lives, saves and consumes over just one period. Moreover, each generation cares about the consumption of the following generations, in the sense that it wants to leave a bequest to the successors. Therefore, such a generation derives utility from its own consumption and those of its descendants. The next generation’s inheritance or capital is described by a production function that is linear with respect to the invested capital. The various versions of this model were studied by numerous authors. For instance, Leininger [52] and Bernheim and Ray [14] independently

proved the existence of a Markov perfect equilibrium for a bequest game with a continuous and increasing production function. Moreover, they assumed that each generation cares only about its own consumption and that of its immediate successor. The reader interested in the existence of Markov perfect equilibria for other specific game examples is referred to Chapter 13 in Fudenberg and Tirole [31], Haurie [42,43], Karp [50] and to Hori [45] that extends the bequest game to an OLG model.

Some stochastic variants of the bequest game are examined in Alj and Haurie [2], Amir [3] and Nowak [62] under specific assumptions imposed on the production function. Their results were recently extended by Balbus and Nowak [6], Nowak [64] and Balbus et al. [8,9] to other more general classes of games, but with standard additive utility aggregator. As mentioned, a Markov perfect equilibrium obtained in Harris and Laibson [40] can also be included as a significant result to the intergenerational stochastic game theory, because the existence theorem, for both the hyperbolic consumer model and the intergenerational game, has exactly the same mathematical meaning.

In this paper, as stated earlier, we study an intergenerational stochastic game, in which each generation can employ the exponential aggregator function. Our model deals with a Borel state space and compact metric action spaces. We enforce some mild continuity conditions on the instantaneous utility and transition functions. Despite the imposed assumptions, proving existence of a stationary Markov perfect equilibrium is much more demanding than in the standard additive utility case (i.e., with the linear aggregator function). This is because the best response correspondence involved in the proof of existence need not be convex-valued. Therefore, one cannot apply a fixed point argument. This situation occurs even in simple cases, i.e., if the state and action spaces are intervals of the set of real numbers and the instantaneous utility and production functions are strictly concave. To circumvent this predicament in a unified setup, we assume that the transition probability function is a convex combination of finitely many atomless measures on the state space with coefficients that may depend on the state and action variables. Such a form of the transition probability has been already utilised in the theory of Markov games by Amir [4], Curtat [22], Nowak [61,63], Horst [46]. Our proof of existence of stationary Markov perfect equilibria consists of two steps. Firstly, we show that there exists a possibly randomised stationary Markov perfect equilibrium. Secondly, making use of the specific structure of the transition probability and applying the theorem of Dvoretzky, Wald and Wolfowitz [27] we obtain a desired pure stationary Markov perfect equilibrium. The former result in contrast to the latter one is only of some technical flavour. Moreover, as a byproduct, we provide a new existence theorem for intergenerational stochastic game with additive expected utility. It is worth emphasising that a pure stationary Markov equilibrium can be obtained without the assumption of atomless measures. But it is only possible at some extra cost. Namely, the state and action spaces have to be intervals of the set of real numbers, the instantaneous utility and production functions ought to be concave, and the measures involved in the transition probability must satisfy a certain stochastic dominance condition. These requirements, however, are sufficient to the study of the additive utility case only. If, on the other hand, one wishes to deal with the exponential aggregator, additional assumptions are expected. These specific assumptions together with results are contained in Section 6. They stress out the fact that the application of the theorem of Dvoretzky, Wald and Wolfowitz [27] is a successful approach in the analysis of stationary Markov equilibria as long as we are concerned with atomless transitions. This observation applies to both cases: with the linear and exponential way of aggregation. Finally, we would like to emphasise that the purification method based upon the theorem of Dvoretzky, Wald and Wolfowitz [27] has not been exploited in the investigation of Markov perfect equilibria in OLG models.

This paper is organised as follows. In Section 2, we discuss certain relationships of our methods with the ideas used in the theory of standard Markov games. Section 3 contains essential definitions and facts such as a version of the theorem of Dvoretzky, Wald and Wolfowitz [27]. Subsection 3.1 is devoted to basic notions concerning risk sensitive optimisation and the utility description for risk sensitive generations. Subsection 3.2 presents some examples on stationary Markov perfect equilibria in simple bequest stochastic games. A careful analysis of these examples can considerably help in understanding the difference between the exponential and linear manner of aggregation. Section 4 contains a description of our general model and main results, whose proofs are included in Appendix A. Furthermore, likewise in Duffie et al. [24], we give conditions under which a stationary Markov perfect equilibrium induces a stationary distribution of the underlying process. This analysis is incorporated in Section 5. Specific results on the existence of stationary Markov perfect equilibria for games, in which transition probabilities may have atoms are given in Section 6. Finally, Section 7 contains many concluding remarks.

2. A comparison of literature on equilibria for standard Markov games

By a standard Markov game we mean a discounted stochastic game with simultaneous moves played on a Borel state space by finitely many players. Such games were extensively examined during the last four decades and received a lot of special attention from both mathematicians and economists. Therefore, a natural question arises as to whether the results and methods utilised in the theory of standard Markov games can be applied to the stochastic bequest games under consideration in this paper. Although, the answer is negative, there is a strong link between the tools and techniques used in the two frameworks. Let us first mention that pretty general results on existence of subgame perfect Nash equilibria in standard Markov games were proved by Mertens and Parthasarathy [57], Solan [70] and Maitra and Sudderth [55]. However, this sort of equilibria were slightly criticised by Maskin and Tirole [56]. They provide a whole set of motivations for studying stationary Markov perfect equilibria rather than subgame perfect ones. Such a discussion is especially helpful and desirable in the context of the study of intergenerational stochastic games. The existence of stationary Markov equilibria in standard Markov games has been examined by a number of authors. Nonetheless, the existence of pure equilibria requires other additional conditions imposed on the instantaneous utility function and the transition probability such as concavity or supermodularity. Moreover, the state and action spaces should be intervals of a Euclidean space. Although, some of the requirements are justified in certain group of models and even possess an economic interpretation, they are usually pretty specific. The problem beyond rests upon finding a fixed point in an appropriate function space, for example, in the class of monotonic Lipschitz functions with bounded modulus. Therefore, for instance, Amir [4] and Curtat [22] utilise the theory of supermodular functions, developed in Topkis [73], and Horst [46], in his analysis, applies Lipschitz selections of best reply functions. Other function spaces were exploited in the papers of Dutta and Sundaram [26] and Nowak [63], where the players are allowed to use discontinuous strategies. In Section 6, we allow the transitions to have atoms. Our assumptions here are akin to the conditions accepted in Curtat [22] or Nowak [63]. Such conditions have been already used in Amir [3], Nowak [62], Balbus and Nowak [6], Balbus et al. [7–9] in the study of intergenerational stochastic games within the expected additive utility framework. Since in this paper we study more involved structure of the game (with the exponential aggregator function), we

encounter further technical difficulties, and consequently, the above methods cannot be directly applied in our setup.

Duffie et al. [24] and Nowak and Raghavan [65] studied Markov games on general state space under much weaker assumptions than the ones exploited in the aforementioned papers. A common feature of these two works is that the transition probability has a density function. Then, making use of a certain coordination mechanism, it is shown the existence of a stationary Markov perfect equilibrium. More precisely, the players can coordinate their mixed actions using “public signals”, and the obtained stationary Markov equilibrium is an equilibrium found in the class of correlated strategies of the players. A related theorem for games with weakly continuous transition probabilities was proved by Harris et al. [41]. Recently, Duggan [25] extended the results of Nowak and Raghavan [65] in the sense that he allowed the instantaneous utility functions to depend on the noisy parameter. His concept, however, has no application to intergenerational stochastic games, because generations choose actions in different periods of time.

A class of standard Markov games that possesses a mixed stationary Markov perfect equilibrium was pointed out by Nowak [61]. His result does not require any concavity and supermodularity conditions. He instead assumes that the transition probability function is a convex combination of finitely many probability measures on the state space with coefficients that may depend on the state and action variables. In addition, these coefficients and utility functions have to be continuous on the product of players’ action sets. This class also embraces the game model, considered by Parthasarathy and Sinha [66], with state independent transition probabilities. The key idea of the proof in Nowak [61] rests upon the Lyapunov theorem on the range of atomless vector measures. More precisely, by Lyapunov’s theorem one can select appropriate extreme points in the space of convex combinations of Nash equilibrium payoffs of the players in some auxiliary one shot game. Then, a measurable implicit function theorems are allowed to obtain Nash equilibrium strategies. The assumptions made in Nowak [61] are stronger than in Nowak and Raghavan [65]. However, as Levy [53] showed recently, stationary Markov perfect equilibria may not exist in discounted dynamic games with deterministic transitions.

In this paper we make a similar assumption on the transition probability (stochastic production function) to that of Nowak [61]. The advantage of such a framework is the fact that we deal with a Borel state space and do not require further limitations imposed on the model except for standard continuity conditions with respect to the players’ actions. In order to get a stationary Markov perfect equilibrium in the bequest game under consideration we allow the players to randomise their actions. First, as a technical result, we show the existence of a randomised equilibrium, and then we apply a purification technique, exploited extensively in statistical decision theory by Dvoretzky, Wald and Wolfowitz [27]. This method of purification also relies on Lyapunov’s theorem on the range of atomless vector measures,¹ but it completely differs from the approach taken in Nowak [61]. Namely, in our paper we are not concerned with the payoff realisations, but we directly replace mixed strategies by suitable chosen pure ones. In this manner, we avoid numerous technical assumptions usually made in the theory of intergenerational games. Moreover, the application of the theorem of Dvoretzky, Wald and Wolfowitz [27] is crucial in the study of game models with the exponential aggregator function, because the best response mappings may not be convex valued, even in the cases with one dimensional state space. We refer the

¹ See Dvoretzky, Wald and Wolfowitz [27], Feinberg and Piunovskiy [28] or Feinberg and Piunovskiy [29].

reader to further detailed comments that are provided following a model description and result formulations.

3. Preliminaries

3.1. Risk-sensitive optimisation

Expected utility theory introduced by von Neumann and Morgenstern, and then criticised and challenged over sixty years remains the benchmark economic approach. Von Neumann and Morgenstern showed that if a person accepts certain set of assumptions concerning “rational choice”, then he/she should compare alternatives by use of the expected utility calculations. The utility function chosen by an individual should reflect to some extent his/her risk tolerance. For example, a reasonable utility, say U , ought to be increasing (more wealth is better than less) and concave (the marginal utility of wealth is decreasing). If U is concave everywhere, then the individual who employs it, is called risk averse. This interpretation follows from the von Neumann and Morgenstern representation of the underlying preference relation. More precisely, let X be a random variable defined on a space Ω endowed with some σ -algebra. X can be regarded as a random payoff in some optimisation model. The symbol E represents expectation with respect to a fixed probability distribution P on Ω . Then, Jensen’s inequality yields that

$$EU(X) \leq U(EX) \quad \text{if } E|X| < +\infty.$$

This means that the sure investment with the same expectation is always preferred by a risk averse decision maker. If the individual grades his/her random payoffs with the aid of the linear function, then he/she is said to be risk neutral, since no account for risk is made.

Let U^r be defined as follows

$$U^r(x) = \begin{cases} -e^{rx}, & \text{for } r < 0 \\ x, & \text{for } r = 0. \end{cases} \quad (1)$$

At this point we would like to remark some attractive properties of U^r . First, the function U^r belongs to the so-called CARA (Constant Absolute Risk Aversion) utilities, which means that U^r has a constant *constant risk sensitivity coefficient* in the sense of Arrow–Pratt measure (index), see Föllmer and Schied [30], Pratt [69]. Let us recall that the Arrow–Pratt index for a twice differentiable utility function U is defined as $-U''(x)/U'(x)$. Clearly, it equals $-r$ for U^r . Second, U^r is strictly increasing and continuous. Moreover, U^r is strictly concave if $r < 0$.² Third, due to the intermediate value theorem there exists a real number $\mathcal{E}(r, X)$ such that $U^r(\mathcal{E}(r, X)) = EU^r(X)$. The value $\mathcal{E}(r, X)$ is called *certainty equivalent* and from (1) it follows that

$$\mathcal{E}(r, X) = \begin{cases} \frac{1}{r} \ln(Ee^{rX}), & \text{for } r < 0 \\ EX, & \text{for } r = 0. \end{cases}$$

Hence, the individual with risk sensitivity coefficient $-r > 0$ is indifferent between receiving a random payoff X and obtaining the amount $\mathcal{E}(r, X)$ for sure. Let us assume now that X does not equal to some constant almost everywhere. Observe that the Taylor expansion around $r = 0$ for U^r yields that

² Note that we do not consider the function U^r with positive values of r , since then the function U^r is convex. Such a utility reflects a risk seeking attitude to risk of the individual that is seldom met in real-life problems.

$$U^r(X) = 1 + \sum_{k=1}^{\infty} \frac{r^k}{k!} EX^k.$$

For the certainty equivalent, we have

$$\mathcal{E}(r, X) = \frac{1}{r} \ln \left(1 + \sum_{k=1}^{\infty} \frac{r^k}{k!} EX^k \right) \approx EX + \frac{r}{2} \text{Var} X,$$

if r is sufficiently close to zero. Therefore, if $r < 0$, then the individual who considers $\mathcal{E}(r, X)$ thinks not only of the expected value EX of the random payoff X , but also of its variance. This certainty equivalent of X is sometimes called the entropic risk measure of X , see e.g., Föllmer and Schied [30], and was already used in stochastic control theory forty years ago by Howard and Matheson [47]. For further results in this area the reader is referred, among others, to Cavazos-Cadena and Fernández-Gauchera [21], Jaśkiewicz [48] and references therein. Nowadays this optimisation criterion has found a number of applications in stochastic optimisation (Whittle [75]), in finance and portfolio management (Bielecki and Pliska [16], Föllmer and Schied [30]), in dynamic games (Başar [13], Monahan and Sobel [58], Caravani and Papavasiliopoulos [20]) and macroeconomics (Hansen and Sargent [38,39], Tallarini [72]).

Furthermore, denote by Λ a set of actions of the decision maker (player) and let $P = P_\lambda$ depend on $\lambda \in \Lambda$. The aforementioned discussion implies that

$$\mathcal{E}_\lambda(r, X) := \frac{1}{r} \ln \int_{\Omega} e^{rX(\omega)} P_\lambda(d\omega) = \frac{1}{r} \ln E_\lambda e^{rX}$$

is a reasonable utility for a risk averse player. We would like to point out that maximisation of $\mathcal{E}_\lambda(r, X)$ with respect to λ is equivalent to maximisation of $\int_{\Omega} (-e^{rX(\omega)}) P_\lambda(d\omega)$. In the sequel, we shall often use this observation.

For better understanding this optimisation criterion, let us consider the following simple example. Assume that a person can choose either action a_0 that results in obtaining 0 for sure, or action a_λ that gives a random payoff X with the distribution $P_\lambda(X = 1) = \lambda$ and $P_\lambda(X = -1) = 1 - \lambda$, where $\lambda \in (0, 1)$. If the decision maker is risk neutral (equipped with a linear utility function), then he/she accepts the lottery if $\lambda \geq 1/2$. If, on the other hand, the individual is risk averse and grades his/her random payoffs according to the function U^r ($r < 0$), then he/she is willing to accept the lottery than a sure amount if $0 \leq \frac{1}{r} \ln E_\lambda e^{rX}$, which implies that $\lambda \geq \lambda_r := 1/(e^r + 1)$. If $\lambda = \lambda_r$, then the decision maker is indifferent between taking part in the lottery and obtaining nothing. Moreover, observe that

$$\lim_{r \rightarrow 0^-} \lambda_r = \frac{1}{2} \quad \text{and} \quad \lim_{r \rightarrow -\infty} \lambda_r = 1.$$

We may also take a look at λ_r in terms of the expected value of the random payoff. Clearly, $E_\lambda X = 2\lambda - 1$ and $\lambda \geq \lambda_r$ implies that $E_\lambda X \geq \frac{1-e^r}{1+e^r}$. In particular, this means that for $r = -1$ the individual decides to take part in the lottery, if its expected value equals at least 0.462117. The variance of X with respect to P_{λ_r} equals $\text{Var}_{\lambda_r} X = 4(\lambda_r - \lambda_r^2) = \frac{4e^r}{(1+e^r)^2}$ and it is easy to notice that $\text{Var}_{\lambda_r} X \rightarrow 0$ if $r \rightarrow -\infty$, i.e., $\lambda_r \rightarrow 1^-$. It is obvious that a lottery with $\lambda \approx 1$ may not exist in real-life models, but the decision maker can choose his/her risk sensitivity coefficient that uniquely describes his/her the threshold λ_r at which he/she starts to appreciate the lottery than the sure amount.

In the game model described later and studied in this paper we consider the composition of U^r with some concave functions called partial utilities of generations. At this point we wish to emphasise that this approach enables us to generate various classes of von Neumann and Morgenstern utilities. More precisely, assume that the decision maker employs his/her partial utility function w , which is twice differentiable function, and $w' > 0$, and $w'' < 0$. Then, he/she deals with the function $U^r \circ w$. Note that $U^r \circ w$ is concave, and therefore we are still considering the von Neumann and Morgenstern utility. Furthermore, we observe that the Arrow–Pratt measure of absolute risk aversion for $U^r \circ w$ need not be constant and equals

$$R(x) := -\frac{(U^r \circ w)''(x)}{(U^r \circ w)'(x)} = -rw'(x) - \frac{w''(x)}{w'(x)},$$

where the second term is the risk sensitivity coefficient of w . Clearly, if the function $x \mapsto -\frac{w''(x)}{w'(x)}$ is decreasing or constant, then R is either decreasing or constant, which means that $U^r \circ w$ is either DARA (Decreasing Absolute Risk Aversion) or CARA utility. It is easily to see that the Arrow–Pratt index for $U^r \circ w$ equals 0 if and only if $r = 0$ and w is an affine function. Assume that $r < 0$ and let us take a look at typical examples of w (see, for instance, Föllmer and Schied [30]).

- If $w(x) = ax$, with $a > 0$, then $R(x) = -ra$ and $U^r \circ w$ is again CARA utility.
- If $w(x) = \ln x$ on $(0, \infty)$, then $R(x) = \frac{1}{x}(1 - r)$ and $U^r \circ w$ is HARA (Hyperbolic Absolute Risk Aversion) and DARA (Decreasing Absolute Risk Aversion) utility.
- If $w(x) = \frac{x^{1-\rho}-1}{1-\rho}$ on $(0, \infty)$ with $\rho \neq 1$ and $\rho > 0$, then $R(x) = -rx^{-\rho} + \rho\frac{1}{x}$ and $U^r \circ w$ is DARA utility.
- If $w(x) = -e^{\alpha x}$, with $\alpha < 0$, then $R(x) = -\alpha + r\alpha e^{\alpha x}$ and $U^r \circ w$ is DARA utility.

In this manner we are able to obtain various DARA utilities, which reflect the more common behaviour of the individual than IARA (Increasing Absolute Risk Aversion) utilities.

3.2. Risk-sensitive bequest games: examples

In this subsection we consider a stochastic version of the bequest game. Similar examples in the additive setting ($r = 0$) were studied, for instance, in Amir [3], Balbus et al. [7]. We assume that there is a single good used both for a consumption and as productive capital. The quantity of the good is described by a number from the interval $S := [0, 1]$. Generation t ($t = 1, 2, \dots$) lives for one period (period t) and inherits an amount of good $s_t \in S$ from generation $t - 1$. Generation t consumes $0 \leq a_t \leq s_t$ and saves the remaining part $s_t - a_t$. The next generation inheritance is s_{t+1} , where s_{t+1} is drawn from S according to the distribution

$$q(\cdot | s_t, a_t) = (1 - (s_t - a_t))\delta_0(\cdot) + (s_t - a_t)\nu(\cdot).$$

Here $\delta_0(\cdot)$ is the Dirac measure concentrated at point 0 and $\nu(\cdot)$ is some probability measure on S . Furthermore, the partial utility of the current generation is $u(a) = a$, whereas the successor's partial utility is $v(a) = 4a$. We assume that the current generation aggregates the two above mentioned partial utilities by means of the function U^r . Since we consider a very special case of our general model from next sections, we shall give only standard and necessary definitions. Let F be the set of all measurable functions $f : S \mapsto S$ such that $f(s) \in [0, s]$ for each $s \in S$.

Elements of F are called (consumption) strategies for generations. If the current generation consumes $a \in [0, s]$ and the follower is assumed to use a strategy $f \in F$, then the expected payoff for the current generation is

$$E(s, a, f, U^r) := \int_0^1 U^r(u(a) + v(f(s')))q(ds'|s, a).$$

A pure *Stationary Markov Perfect Equilibrium* (SMPE for short) in the bequest game is a function $f^* \in F$ such that

$$f^*(s) \in \arg \max_{a \in [0, s]} E(s, a, f^*, U^r)$$

for all $s \in S$. Below in our calculations, we shall uniquely specify a number $s_r \in S$ for each $r \leq 0$. With this number we shall associate a strategy f_r defined as follows

$$f_r(s) = \begin{cases} s, & \text{for } s \leq s_r \\ 0, & \text{for } s > s_r. \end{cases} \quad (2)$$

We now give two examples and calculate SMPE for $r = 0$ and $r < 0$.

Example 1. Let $v(\cdot)$ be a uniform distribution on S and $s \in S$ be a capital of the current generation.

(I) If $r = 0$ and the follower employs a strategy f , then the current generation faces the following problem

$$\max_{a \in [0, s]} E(s, a, f, U^0) = \max_{a \in [0, s]} (a + (s - a)j_0), \quad \text{where } j_0 = \int_0^1 4f(y)dy. \quad (3)$$

Note that $f^* = f$ is a SMPE if and only if $j_0 = 1$. Thus, there are infinitely many SMPE. For example, $f^*(s) = s/2$ for all $s \in S$. Assume now that the successor uses a strategy f_0 defined in (2) for some $s_0 \in S$. Then, equation $j_0 = \int_0^{s_0} 4ydy = 1$ implies that $2s_0^2 = 1$. Hence, $s_0 = \sqrt{2}/2 \approx 0.707107$. Thus, among infinitely many SMPE we can distinguish one $f^* = f_0$ of the form (2).

(II) Assume now that $r < 0$ and the successor employs some strategy f . The current generation faces the following problem

$$\max_{a \in [0, s]} E(s, a, f, U^r) = \max_{a \in [0, s]} (-e^{ra}[1 + (s - a)(j_r - 1)]),$$

where $j_r = \int_0^1 e^{4rf(y)}dy$. Clearly, the above display is equivalent to

$$\min_{a \in [0, s]} e^{ra}[1 + (s - a)(j_r - 1)].$$

It is not difficult to observe that the function $a \mapsto e^{ra}[1 + (s - a)(j_r - 1)]$ (defined for $a \in [0, s]$) has the minimum either at $a = 0$ or at $a = s$ regardless of the form of f (more precisely, the value of $j_r \in [0, 1]$). Hence, $e^{rs} < 1 + s(j_r - 1)$ or $e^{rs} \geq 1 + s(j_r - 1)$ for $s \in S$. Let us now consider $f = f_r$ defined in (2). Clearly, from the above discussion, it follows that $f^* = f_r$ is a SMPE if s_r is a solution of the following equation

$$e^{rs_r} = 1 + s_r(j_r - 1). \quad (4)$$

Note that $j_r = \frac{e^{4rs_r}}{4r} - \frac{1}{4r} - s_r$ and Eq. (4) is equivalent to the following

$$\frac{e^{rs_r}}{s_r} - \frac{1}{s_r} - \frac{e^{4rs_r}}{4r} + \frac{1}{4r} + s_r = 0. \quad (5)$$

It is easy to see that this equation has a unique solution in S . For instance, the solutions are $s_{-0.001} \approx 0.707315$, $s_{-1} \approx 0.902023$, and $s_{-2} \approx 0.993497$. It can be also concluded that $f^* = f_r$ is a unique left continuous *SMPE* in this game.

Observe that the function $a \mapsto E(-\exp[r(u(a) + v(f(s'))))]$ is neither convex nor concave. Moreover, if $f = f^* = f_r$, then $\arg \max_{a \in [0, s_r]} E(s_r, a, f_r, U^r) = \{0, s_r\}$. These facts indicate that an analysis of *SMPE* for the case $r < 0$ is more complicated than in the additive case, in which the functions involved can be concave, see for example, Balbus et al. [7,8].

Example 2. Let now $v(\cdot)$ have the density function $p(y) = 2y$, $y \in S$.

(I) Assume that $r = 0$. Clearly, if f is used by the next generation, then the current generation faces the problem as in (3) but with $j_0 = \int_0^1 8yf(y)dy$. We note that any function $f^* = f$ for which $j_0 = 1$ is again a *SMPE*. We now consider f_0 in (2). Then, $f^* = f_0$ is a *SMPE*, if $\int_0^{s_0} 8y^2 dy = 1$. Hence, $s_0 = \sqrt[3]{3/8} \approx 0.721125$.

(II) Let $r < 0$. Similarly as in Example 1, we can observe that in this case there is a unique left continuous *SMPE* and it is of the form (2) with suitably chosen s_r . Assuming that the following generation is going to use f_r we can show that the current generation can reply optimally by playing the same strategy provided that s_r solves Eq. (4), where

$$j_r = \int_0^1 e^{4rf_r(y)} 2y dy = \frac{e^{4rs_r s_r}}{2r} - \frac{e^{4rs_r}}{8r^2} + \frac{1}{8r^2} + 1 - s_r^2.$$

Summing up, Eq. (4) takes on the form

$$\frac{e^{rs_r}}{s_r} - \frac{1}{s_r} - \frac{e^{4rs_r s_r}}{2r} + \frac{e^{4rs_r}}{8r^2} - \frac{1}{8r^2} + s_r^2 = 0. \quad (6)$$

The examples of solutions are $s_{-0.01} \approx 0.722859$, $s_{-1} \approx 0.879446$, and $s_{-4} \approx 0.996388$.

Let us discuss the *SMPE* given by (2). If the current generation uses the exponential function to aggregate partial utilities, then the changing point s_r for consumption level (from the whole stock to zero) is greater than in case of the linear aggregator U^0 . Note further that from the definition of the transition law, it follows that the higher stock is, the greater probability of the stock level for the follower is. Since the current generation derives its utility from the partial utilities u and v , then it is willing to leave the whole stock for the next generation only if the probability of the future level of stock is sufficiently large. We may also look at the expected value and the variance of random variables: X_1 having the uniform distribution on $[0, 1]$ and X_2 having the density function p from Example 2. Namely, $EX_1 = 1/2$, $Var X_1 = 1/12$ and $EX_2 = 2/3$, $Var X_2 = 1/18$. Hence, it is not peculiar that s_r in Example 1 is greater than in Example 2. This fact means that the current generation does not care only about the expected value of the future stock, but also about its volatility. Moreover, we observe in both examples that if r is sufficiently small, i.e., $r \approx -2.154$ in Example 1 and $r \approx -5.483$ in Example 2 (the Arrow–Pratt index $-r$ is high), then the unique *SMPE* is to consume the whole capital. If, on the other hand, $r \rightarrow 0^-$, then $s_r \rightarrow s_0$. Indeed, it is enough to use the Taylor expansion of the

function $z \mapsto e^z$ around point 0. For example, making use of $e^z \approx 1 + z + z^2/2$ for $z = rs_r$ and $z = 4rs_r$ we obtain from (5) that

$$1 + r \frac{s_r}{2} - 2s_r^2 \approx 0 \quad \text{and} \quad s_r \rightarrow \sqrt{2}/2 \quad \text{if } r \rightarrow 0^-.$$

Similarly, applying $e^z \approx 1 + z + z^2/2 + z^3/6$ for $z = rs_r$ and $z = 4rs_r$ to (6) it follows that

$$\frac{16r}{3}s_r^4 + \frac{8}{3}s_r^3 - \frac{r^2}{6}s_r^2 - \frac{r}{2}s_r - 1 \approx 0 \quad \text{and} \quad s_r \rightarrow \sqrt[3]{\frac{3}{8}} \quad \text{if } r \rightarrow 0^-.$$

Thus, the considered examples confirm the interesting fact that the stationary Markov perfect equilibria of the form f_r (unique for every r) converge to a *SMPE* of the form f_0 as $r \rightarrow 0^-$.

3.3. Basic notions and relevant facts

Let R be the set of all real numbers, $R_+ := [0, \infty)$. Let Y be a Borel space, i.e., a non-empty Borel subset of a complete separable metric space endowed with its Borel σ -algebra $\mathcal{B}(Y)$. For any Borel space X , by $P(X)$ we denote the space of all probability measures on X endowed with the weak topology and the Borel σ -algebra, see Chapter 7 in Bertsekas and Shreve [15]. Let S and A be Borel spaces. A *transition probability* or a *stochastic kernel* from S to A is a function $\psi : \mathcal{B}(A) \times S \mapsto [0, 1]$ such that $\psi(B|\cdot)$ is a Borel measurable function on S for every $B \in \mathcal{B}(A)$ and $\psi(\cdot|s) \in P(A)$ for each $s \in S$. It is well-known that every Borel measurable mapping $f : S \mapsto P(A)$ induces a transition probability ψ from S to A . Namely, $\psi(B|s) = f(s)(B)$, $B \in \mathcal{B}(A)$, $s \in S$, see Proposition 7.26 in Bertsekas and Shreve [15]. We shall usually write $\psi(da|s)$ instead of $f(s)(da)$. Clearly, any Borel measurable mapping $f : S \mapsto A$ is a special transition probability φ from S to A such that for each $s \in S$, $\varphi(\cdot|s)$ is the Dirac measure concentrated at the point $f(s)$.

Let C be a Borel subset of $S \times A$ such that the set

$$A(s) := \{a \in A : (s, a) \in C\} \neq \emptyset$$

for each $s \in S$. In addition, assume that for every $s \in S$ the set $A(s)$ is compact. By Brown and Purves [19], the set-valued mapping (correspondence) $s \mapsto A(s)$ admits a Borel selector, that is, there exists a Borel measurable mapping $f : S \mapsto A$ such that $f(s) \in A(s)$ for all $s \in S$. Let F be the set of all such Borel selectors. By Ψ we denote the set of all transition probabilities ψ from S to A such that $\psi(A(s)|s) = 1$ for each $s \in S$. Clearly, $F \subset \Psi$, so $\Psi \neq \emptyset$.

Subsequently, we shall use a version of a classical result on elimination of randomisation in statistical decision theory, see Dvoretzky, Wald and Wolfowitz [27], Balder [11]. Related results were utilised in the theory of controlled stochastic processes by, for instance, Feinberg and Piunovskiy [28,29].

Let $\mu_1, \dots, \mu_l$ be probability measures on S . Consider a family $w_1, \dots, w_m$ of real-valued Borel measurable functions on C . For a proof of the following lemma, consult Theorem 1 in Feinberg and Piunovskiy [29] and Theorem 2.1 in Feinberg and Piunovskiy [28].

Lemma 1. Assume that the measures $\mu_1, \dots, \mu_l$ are atomless and $\hat{w}_1, \dots, \hat{w}_m$ are non-negative Borel measurable functions on S such that $\int_S \hat{w}_j(s) \mu_k(ds) < \infty$ for all $j = 1, \dots, m$, $k = 1, \dots, l$. Assume also that $|w_j(s, a)| \leq \hat{w}_j(s)$ for each $j = 1, \dots, m$ and $(s, a) \in C$. Suppose that $A^*(s)$ is a non-empty compact subset of $A(s)$ for each $s \in S$ and the set $\{(s, a) : s \in S, a \in A^*(s)\}$

is Borel. Then for each $\psi \in \Psi$ such that $\psi(A^*(s)|s) = 1$ for all $s \in S$, there exists some $f \in F$ such that $f(s) \in A^*(s)$ for each $s \in S$ and

$$\int_S \int_{A(s)} w_j(s, a) \psi(da|s) \mu_k(ds) = \int_S w_j(s, f(s)) \mu_k(ds)$$

for all $j = 1, \dots, m$, $k = 1, \dots, l$.

4. The general model and main results

In this section, we study intergenerational games with a stochastic production function and a Borel state space. Our model is a generalisation of numerous consumption-saving models and stochastic production economies with capital and labour.

4.1. Finitely many descendants

We assume that time is discrete and is indexed by $t \in T = \{1, 2, \dots\}$. We consider an *intergenerational stochastic game* played by a countable family $\{i_t\}_{t \in T}$ of short-lived players (generations). It is defined by the objects: S , A , C , $\{A(s)\}_{s \in S}$, u , $v_1, \dots, v_m$, and q , where:

- (A1) S is a Borel state space.
- (A2) A is a Borel space of actions available to every player i_t . For any $s \in S$, $A(s)$ is a non-empty compact subset of A representing the set of all actions available to player i_t in state $s \in S$. It is assumed that the set

$$C = \{(s, a) : s \in S, a \in A(s)\}$$

is Borel in $S \times A$.

- (A3) $u, v_1, \dots, v_m$ are Borel measurable real-valued functions defined on the set C .
- (A4) q is a transition probability from C to S , called the *law of motion* among states. If s_t is a state at the beginning of period t of the game and player i_t selects an action $a_t \in A(s_t)$, then $q(\cdot | s_t, a_t)$ is the probability distribution of the next state $s_{t+1} \in S$.

The sets Ψ and F are defined as in Subsection 3.3. A *randomised Markov strategy*³ for player (generation) i_t is a function $\psi_t \in \Psi$. Let $\{\psi_t\}_{t \in T}$ be a sequence of randomised Markov strategies of all generations. For any $t \in T$, define

$$\psi^t := \{\psi_\tau : \tau = t, t+1, \dots\}.$$

The game is played in the following way. Generation i_t lives in period t and inherits a state of the economy $s_t \in S$ from the preceding generation i_{t-1} (that lived and consumed in period $t-1$). Next it chooses an action (consumes) $a_t \in A(s_t)$ and derives its partial utility by computing $u(s_t, a_t)$. Moreover, it assumes that each descendant i_{t+j} (for $j = 1, \dots, m$) is equipped with the partial utility function v_j and selects some action $a_{t+j} \in A(s_{t+j})$, where the state s_{t+j} evolves according to the transition law q . Then, the current generation considers the sum of $m+1$ partial

³ Allowing for randomised strategies we are able to prove an existence theorem for Markov perfect equilibria in a pretty general setup. Pure equilibria are then obtained using a purification result of Dvoretzky, Wald and Wolfowitz [27].

utilities $u(s_t, a_t) + v_1(s_{t+1}, a_{t+1}) + \dots + v_m(s_{t+m}, a_{t+m})$. Clearly, this sum is a random variable defined on $\Omega := H_m^t$, where

$$H_m^t := A(s_t) \times C \times \dots \times C \quad (C \text{ is taken } m \text{ times})$$

is the space all feasible histories of the process from the state s_t endowed with the product σ -algebra. Assume now that all generations employ a randomised Markov strategy ψ_τ , $\tau \geq t$, defined as above. According to the Ionescu-Tulcea Theorem (see Proposition V.1.1 in Neveu [60] or Chapter 7 in Bertsekas and Shreve [15]) for each $s_t \in S$ there exists a unique probability measure $P_{s_t}^{\psi^t}$ defined on $\Omega = H_m^t$ induced by ψ^t and the transition probability q . Denote by $E_{s_t}^{\psi^t}$ the expectation operator with respect to the probability measure $P_{s_t}^{\psi^t}$. Then, generation t calculates the certainty equivalent of the sum of partial utilities aggregated by the function U^r defined in (1), that is, it considers

$$\begin{cases} \frac{1}{r} \ln E_{s_t}^{\psi^t} e^{(ru(s_t, a_t) + r \sum_{j=1}^m v_j(s_{t+j}, a_{t+j}))}, & \text{for } r < 0 \\ E_{s_t}^{\psi^t} \left(ru(s_t, a_t) + \sum_{j=1}^m v_j(s_{t+j}, a_{t+j}) \right), & \text{for } r = 0. \end{cases} \quad (7)$$

Clearly, (7) is well-defined if the functions $u, v_1, \dots, v_m$ are bounded. Subsequently, we shall make some integrability assumptions that allow them to be unbounded. Within such a framework the random variable X in Subsection 3.1 equals to $u(s_t, a_t) + \sum_{j=1}^m v_j(s_{t+j}, a_{t+j})$. Observe that (7) in case of $r < 0$ is equivalent to the study of

$$W_m^r(\psi^t)(s_t) := E_{s_t}^{\psi^t} U^r \left(u(s_t, a_t) + \sum_{j=1}^m v_j(s_{t+j}, a_{t+j}) \right). \quad (8)$$

Hence, we are concerned with von Neumann and Morgenstern utility approach. By putting $V_j(s, a) := e^{rv_j(s, a)}$ for $(s, a) \in C$, $j = 1, \dots, m$, and using (1), we point out that for $r < 0$, we have

$$U^r \left(u(s_t, a_t) + \sum_{j=1}^m v_j(s_{t+j}, a_{t+j}) \right) = -e^{ru(s_t, a_t)} V_1(s_{t+1}, a_{t+1}) \dots V_m(s_{t+m}, a_{t+m}). \quad (9)$$

Obviously, if $r = 0$, we obtain

$$W_m^0(\psi^t)(s_t) := E_{s_t}^{\psi^t} \left(u(s_t, a_t) + \sum_{j=1}^m v_j(s_{t+j}, a_{t+j}) \right). \quad (10)$$

From (9) and (10), it follows that along any trajectory each generation i_t (that lives one period) derives its utility from the current state-action pair (s_t, a_t) and the actions a_{t+j} that will possibly be taken in future states s_{t+j} ($j = 1, \dots, m$) by the m following generations. In terms of consumption and saving models, each generation derives its utility from its own choice and consumption decisions of the m descendants. In the literature, it is often assumed that $v_j := \alpha \beta^j u$, where $\beta \in (0, 1)$ is a long-term discount rate and $\alpha > 0$ is an altruism factor towards future generations, see Phelps and Pollak [68], Alj and Haurie [2], Nowak [64] and references cited therein. We write $W_m^r(\psi)(s_t)$ for $W_m^r(\psi^t)(s_t)$ when all generations i_τ , $\tau \geq t$, use the same strategy $\psi \in \Psi$. Below, with the aid of certain operators, we provide a more legible form for

$W_m^r(\psi)(s_t)$. This form will also be utilised in our proofs. Let $\psi \in \Psi$ and $\nu \in P(A(s))$. Define the transition probability functions induced by q , ψ and ν as follows

$$q(D|s, \psi) := \int_{A(s)} q(D|s, a) \psi(da|s), \quad q(D|s, \nu) := \int_{A(s)} q(D|s, a) \nu(da), \quad D \in \mathcal{B}(S).$$

If $f \in F$ and $s \in S$, then $q(D|s, f) := q(D|s, f(s))$, $D \in \mathcal{B}(S)$. Furthermore, for any Borel function $w : C \mapsto R$, and $\psi \in \Psi$, we put

$$w(s, \psi) := \int_{A(s)} w(s, a) \psi(da|s), \quad s \in S.$$

Additionally, for any Borel function $v : S \mapsto R$ and $\psi \in \Psi$, we set

$$\mathcal{Q}_\psi^{(1)} v(s) = \mathcal{Q}_\psi v(s) := \int_S v(s') q(ds'|s, \psi), \quad \mathcal{Q}_\psi^{(t+1)} v(s) := \mathcal{Q}_\psi^{(1)} \mathcal{Q}_\psi^{(t)} v(s), \quad s \in S, \quad (11)$$

and for $j = 1, \dots, m$, we define

$$(V_j \mathcal{Q})_\psi v(s) := \int_{A(s)} \int_S v(s') q(ds'|s, a) V_j(s, a) \psi(da|s), \quad s \in S, \quad (12)$$

provided that these integrals exist. Making use of (12) for $\psi \in \Psi$, $s_{t+1} \in S$ and $r \neq 0$, we define

$$J_m^r(\psi)(s_{t+1}) := (V_1 \mathcal{Q})_\psi \cdots (V_{m-1} \mathcal{Q})_\psi \tilde{V}_m(\psi)(s_{t+1}), \quad (13)$$

where $\tilde{V}_m(\psi)(y) = V_m(y, \psi)$, $y \in S$. In order to make this definition more readable for a potential reader, we provide an integral formula for (13) when $m = 3$:

$$\begin{aligned} J_3^r(\psi)(s_{t+1}) &= \int_{A(s_{t+1})} \int_S \int_{A(s_{t+2})} \int_S \int_{A(s_{t+3})} V_3(s_{t+3}, a_{t+3}) \psi(da_{t+3}|s_{t+3}) q(ds_{t+3}|s_{t+2}, a_{t+2}) \\ &\quad \times V_2(s_{t+2}, a_{t+2}) \psi(da_{t+2}|s_{t+2}) q(ds_{t+2}|s_{t+1}, a_{t+1}) V_1(s_{t+1}, a_{t+1}) \psi(da_{t+1}|s_{t+1}), \end{aligned}$$

where $V_j(s_{t+j}, a_{t+j}) = e^{r v_j(s_{t+j}, a_{t+j})}$, $j = 1, 2, 3$. Using (11), for $r = 0$, any $\psi \in \Psi$ and $s_{t+1} \in S$, we define

$$J_m^r(\psi)(s_{t+1}) = J_m^0(\psi)(s_{t+1}) := v_1(s, \psi) + \sum_{j=2}^m \mathcal{Q}_\psi^{(j-1)} \tilde{v}_j(\psi)(s) \quad (14)$$

where $\tilde{v}_j(\psi)(y) := v_j(y, \psi)$, $j = 1, \dots, m$, $y \in S$.

Suppose that all generations i_τ , $\tau \geq t$, use the same Markov strategy $\psi \in \Psi$. Then $\{\psi^t\}$ can be identified with ψ . The expected utility function (8) to every generation i_t can be expressed with the help of functions (13) and (14) as follows

$$W_m^r(\psi)(s_t) = \begin{cases} - \int_{A(s_t)} e^{ru(s_t, a_t)} \int_S J^r(\psi)(s_{t+1}) q(ds_{t+1}|s_t, a_t) \psi(da_t|s_t), & \text{for } r < 0 \\ \int_{A(s_t)} (u(s_t, a_t) + \int_S J^0(\psi)(s_{t+1}) q(ds_{t+1}|s_t, a_t)) \psi(da_t|s_t), & \text{for } r = 0. \end{cases} \quad (15)$$

We can now define the main equilibrium concept studied in this paper. For any generation i_t and $s_t = s \in S$, consider the optimisation problem $\Gamma(\psi, s)$ where $\psi \in \Psi$ is a Markov randomised strategy used by every generation i_τ , $\tau \geq t + 1$. The objective function for player i_t in this optimisation problem depends on $a \in A(s)$ and is of the form

$$p_m^r(s, \psi)(a) := \begin{cases} -e^{ru(s,a)} \int_S J_m^r(\psi)(s')q(ds'|s, a), & \text{for } r < 0 \\ u(s, a) + \int_S J_m^0(\psi)(s')q(ds'|s, a), & \text{for } r = 0. \end{cases} \quad (16)$$

For any $s \in S$, $\phi, \psi \in \Psi$ define

$$R_m^r(s, \phi, \psi) := \int_{A(s)} p_m^r(s, \psi)(a)\phi(da|s), \quad s \in S. \quad (17)$$

Clearly, $R_m^r(s, \phi, \psi)$ is the payoff to generation i_t , when it uses a randomised strategy ϕ in state $s \in S$ and every successor generation employs a Markov strategy $\psi \in \Psi$. We shall write $R_m^r(s, v, \psi)$, if for given $s \in S$, $\phi(da|s)$ in (17) is replaced by $v(da)$ for any $v \in P(A(s))$.

Definition 1. A strategy $\psi^* \in F$ ($\psi^* \in \Psi$) is a (*randomised*) *stationary Markov perfect equilibrium (SMPE)* in the intergenerational stochastic game, if for every $s \in S$,

$$R_m^r(s, \psi^*, \psi^*) = \sup_{\phi \in \Psi} R_m^r(s, \phi, \psi^*). \quad (18)$$

We remind that C is assumed to be a Borel set and all the sets $A(s)$ are non-empty and compact. The following assumptions are fundamental for our main results.

(A5) It is assumed that q has the following form

$$q(B|s, a) = \sum_{k=1}^l g_k(s, a)\mu_k(B), \quad B \in \mathcal{B}(S),$$

where the functions $g_k : C \mapsto [0, 1]$ are Borel measurable, $g_k(s, \cdot)$ is continuous on $A(s)$ for each $s \in S$, $k = 1, \dots, l$, and

$$\sum_{k=1}^l g_k(s, a) = 1 \quad \text{for all } (s, a) \in C.$$

Let

$$\mu := \frac{1}{l} \sum_{k=1}^l \mu_k.$$

Clearly, $q(\cdot|s, a) \ll \mu$ for each $(s, a) \in C$.

(A6) It is assumed that $u(s, \cdot)$, and $v_1(s, \cdot), \dots, v_m(s, \cdot)$ are continuous on $A(s)$ for each $s \in S$.

(A7) If $r = 0$, it is imposed that

$$\int_S \max_{a \in A(s)} |u(s, a)| \mu_k(ds) < \infty, \quad \int_S \max_{a \in A(s)} |v_j(s, a)| \mu_k(ds) < \infty$$

for all $j = 1, \dots, m$ and $k = 1, \dots, l$.

We have arrived at our main results. Their proofs are given in [Appendix A](#).

Theorem 1. *Under assumptions (A1)–(A7) the intergenerational stochastic game has a randomised SMPE.*

Theorem 2. *Assume (A1)–(A7) and that $\mu_1, \dots, \mu_l$ are atomless. Then, the intergenerational stochastic game has a SMPE.*

Remark 1. Assumption (A5) was used in the theory of standard Markov games in Nowak [61] to prove the existence of a stationary Nash equilibrium. His proof rests upon the Lyapunov theorem on the range of a vector valued atomless measure. In this paper, we employ an advance result (based also on Lyapunov's theorem) due to Dvoretzky, Wald and Wolfowitz [27]. This fact enforces certain limitations on the model. Namely, we deal with finitely many descendants and the transition structure indicated in (A5). However, at the cost of these constraints, we are able to make pretty weak regularity assumptions on the utility functions. In addition, our model is concerned with a general state space. Therefore, the OLG model with capital and labour studied, for instance, by Balbus et al. [7], or an environmental growth model with a pollution externality examined in Balbus et al. [9] can be viewed as special cases in our study.

The following example shows that SMPE need not exist under assumptions of [Theorem 1](#).

Example 3. Assume that $r = 0$. Let $S = \{1, 2\}$, $m = 1$, $A(1) = \{1, 2\}$, $A(2) = \{1\}$ and $u \equiv v_1$. Assume further that state $s = 2$ is absorbing and $u(2, 1) = 0$. In state $s = 1$, we have that $q(1|1, 1) = 1$ and $u(1, 1) = 1$ or $q(2|1, 2) = 1$ and $u(1, 2) = 3$. Hence, there are two Markov strategies $f_1(1|1) = 1$ and $f_2(2|1) = 1$. Let $s = 1$ and $f_1(\cdot|2) = f_2(\cdot|2) = 1$. If the descendant plays f_1 , then the best reply for the current generation is to play f_2 and obtain

$$u(1, f_2(1)) + \sum_{s'=1}^2 v_1(s', f_1(s'))q(s'|1, f_2(1)) = 3 + 0 = 3.$$

If, on the other hand, the successor generation employs f_2 , then the best reply for the current generation is to use f_2 and get

$$u(1, f_1(1)) + \sum_{s'=1}^2 v_1(s', f_2(s'))q(s'|1, f_1(1)) = 1 + 3 = 4.$$

Hence, there does not exist a SMPE. This fact indicates that assumption (A5) in [Theorem 2](#) on atomless measures is essential. Moreover, it is not difficult to check that $\psi^*(1|1) = \psi^*(2|1) = \frac{1}{2}$ is a randomised SMPE. Indeed, assume that the descendant plays ψ^* and assume that the current generation plays the strategy ψ defined as follows $\psi(1|1) = \alpha = 1 - \psi(2|1)$, $\alpha \in [0, 1]$. Then,

$$u(1, \psi) + \sum_{s'=1}^2 v_1(s', \psi^*)q(s'|1, \psi) = 2\alpha + \alpha + 3(1 - \alpha) = 3,$$

which implies the assertion.

4.2. Infinitely many descendants

In this subsection, we consider an infinite number of descendants for each generation. Moreover, we impose the following assumptions on the model:

(A8) The function u is non-negative and $u(s, \cdot)$ is continuous on $A(s)$ for each $s \in S$.

(A9)

$$\int_S \max_{a \in A(s)} |u(s, a)| \mu_k(ds) < \infty, \quad k = 1, \dots, l.$$

Assuming that $v_j(s, a) = \alpha \beta^j u(s, a)$ for $j = 1, 2, \dots$, $(s, a) \in C$ and u is non-negative, we are actually concerned with the limits of the functions in Subsection 4.1 as $m \rightarrow \infty$. This note enables us to simplify the presentation. Then, the random variable X in Subsection 3.1 equals to $u(s_t, a_t) + \alpha \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u(s_\tau, a_\tau)$, which is now defined on $\Omega := H_\infty^t := A(s_t) \times C^\infty$. We endow H_∞^t with the product σ -algebra. Let ψ_τ be any randomised Markov strategy for each generation i_τ , $\tau \geq t$, $t \in T$. Again the Ionescu-Tulcea Theorem guarantees the existence of a unique probability measure $P_{s_t}^{\psi^t}$ on H_∞^t induced by ψ^t and the transition law q . As discussed earlier, one can deal with the certainty equivalent of X or consider the equivalent (from the maximisation point of view) expected utility function given for i_t as

$$W_\infty^r(\psi^t)(s_t) := E_{s_t}^{\psi^t} U^r \left(u(s_t, a_t) + \alpha \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} u(s_\tau, a_\tau) \right).$$

Since u is non-negative, the expectation above is well-defined. By the Lebesgue monotone convergence theorem, we have

$$W_\infty^r(\psi^t)(s_t) = \lim_{m \rightarrow \infty} W_m^r(\psi^t)(s_t),$$

where $W_m^r(i_t, \psi^t)(s_t)$ is as in (8) with the functions v_j introduced above. Assume now that each generation i_τ , $\tau \geq t+1$, uses a randomised Markov strategy $\psi \in \Psi$. Then, for generation i_t , $s_t = s \in S$, and $\phi \in \Psi$ we can put

$$p_\infty^r(s, \psi)(a) := \lim_{m \rightarrow \infty} p_m^r(s, \psi)(a), \quad R_\infty^r(s, \phi, \psi) := \lim_{m \rightarrow \infty} R_m^r(s, \phi, \psi). \quad (19)$$

The limits in (19) exist by the Lebesgue monotone convergence theorem, since $u \geq 0$. We also have

$$R_\infty^r(s, \phi, \psi) = \int_{A(s)} p_\infty^r(s, \psi)(a) \phi(da|s), \quad s \in S. \quad (20)$$

Obviously, $R_\infty^r(s, \phi, \psi)$ is the payoff to generation i_t when it uses ϕ and each successor generation employs a Markov strategy $\psi \in \Psi$. We write $R_\infty^r(s, v, \psi)$, if for given $s \in S$, $\phi(da|s)$ in (20) is replaced by $v(da)$ for arbitrary $v \in P(A(s))$.

Definition 2. A strategy $\psi^* \in F$ ($\psi^* \in \Psi$) is a (randomised) stationary Markov perfect equilibrium (SMPE) in the intergenerational stochastic game with countably many descendants, if for every $s \in S$,

$$R_\infty^r(s, \psi^*, \psi^*) = \sup_{\phi \in \Psi} R_\infty^r(s, \phi, \psi^*).$$

We can now formulate two new results.

Theorem 3. Assume that (A1)–(A2), (A4)–(A5), (A8) hold. Moreover, assume (A9) when $r = 0$ and u is bounded when $r < 0$. Then, the intergenerational stochastic game with countably many descendants has a randomised SMPE.

The proof is given in [Appendix A](#).

Theorem 4. Assume that $r = 0$ and (A1)–(A2), (A4)–(A5), (A8)–(A9) hold. Moreover, assume that $\mu_1, \dots, \mu_l$ are atomless. Then, the intergenerational stochastic game with countably many descendants has a SMPE.

The proof of [Theorem 4](#) makes use of the theorem of Dvoretzky, Wald and Wolfowitz [27] (stated in [Lemma 1](#)) that holds for finitely many functions and atomless measures. Despite the fact that we deal now with countably many descendants, a functional equation (see (22)) that holds for the case $r = 0$, allows us to apply [Lemma 1](#). Unfortunately, this situation does not repeat for the case with $r < 0$ and we are made to proceed differently. A counterpart of [Theorem 4](#) for $r < 0$ is obtained in [Section 6](#) ([Proposition 3](#)), but under specific assumptions.

For any Borel function $v : S \mapsto R$ integrable with respect to q and $\psi \in \Psi$, define

$$L_\psi v(s) := u(s, \psi) + \beta \int_S v(s') q(ds' | s, \psi), \quad s \in S.$$

Let

$$\hat{J}_\infty^0(\psi)(s) := u(s, \psi) + \sum_{j=1}^{\infty} \beta^j Q_\psi^{(j)} \tilde{u}(\psi)(s), \quad (21)$$

where $\tilde{u}(\psi)(y) := u(y, \psi)$, $y \in S$. Under assumption (A5), all the functions $Q_\psi^{(j)} \tilde{u}(\psi)$ are uniformly bounded, thus (21) is well-defined. Moreover, we note that

$$\hat{J}_\infty^0(\psi) = L_\psi \hat{J}_\infty^0(\psi). \quad (22)$$

Proof of Theorem 4. By [Theorem 3](#), there exists a randomised SMPE $\psi^* \in \Psi$. Using (21) it follows that

$$\begin{aligned} l^*(s) &:= \max_{v \in P(A(s))} \left(u(s, v) + \alpha \beta \int_S \hat{J}_\infty^0(\psi^*)(s') q(ds' | s, v) \right) \\ &= u(s, \psi^*) + \alpha \beta \int_S \hat{J}_\infty^0(\psi^*)(s') q(ds' | s, \psi^*) \end{aligned}$$

for all $s \in S$. Put

$$A^*(s) = \arg \max_{a \in A(s)} \left(u(s, a) + \alpha \beta \int_S \hat{J}_\infty^0(\psi^*)(s') q(ds' | s, a) \right).$$

Under our continuity and compactness assumptions $A^*(s)$ is non-empty and compact. Moreover, $\psi^*(A^*(s)|s) = 1$ for each $s \in S$. We have that

$$l^*(s) = \max_{a \in A^*(s)} \left(u(s, a) + \alpha \beta \int_S \hat{J}_\infty^0(\psi^*)(s') q(ds'|s, a) \right), \quad s \in S.$$

From [Lemma 1](#), we infer that there exists a Borel measurable mapping $f^* \in F$ from S into A such that $f^*(s) \in A^*(s)$ for each $s \in S$, and

$$\int_S L_{\psi^*} \hat{J}_\infty^0(\psi^*)(s') \mu_k(ds') = \int_S L_{f^*} \hat{J}_\infty^0(\psi^*)(s') \mu_k(ds')$$

for all $k = 1, \dots, l$. Under assumption (A5), this implies that

$$\int_S L_{\psi^*} \hat{J}_\infty^0(\psi^*)(s') q(ds'|s, a) = \int_S L_{f^*} \hat{J}_\infty^0(\psi^*)(s') q(ds'|s, a) \quad (23)$$

for all $(s, a) \in C$. Hence,

$$\beta Q_{f^*} L_{\psi^*} \hat{J}_\infty^0(\psi^*) = \beta Q_{f^*} L_{f^*} \hat{J}_\infty^0(\psi^*),$$

and consequently,

$$L_{f^*} L_{\psi^*} \hat{J}_\infty^0(\psi^*) = L_{f^*} L_{f^*} \hat{J}_\infty^0(\psi^*).$$

This fact and [\(22\)](#) imply that

$$L_{f^*} \hat{J}_\infty^0(\psi^*) = L_{f^*} L_{\psi^*} \hat{J}_\infty^0(\psi^*) = L_{f^*}^{(m)} \hat{J}_\infty^0(\psi^*) \quad (24)$$

for all $m \geq 2$. Here, $L_{f^*}^{(m)}$ denotes the composition of L_{f^*} with itself m times. It is easy to check that

$$L_{f^*}^{(m)} \hat{J}_\infty^0(\psi^*)(s) = u(s, f^*(s)) + \sum_{j=1}^{m-1} \beta^j Q_{f^*}^{(j)} \tilde{u}(f^*) + \beta^m Q_{f^*}^{(m)} \hat{J}_\infty^0(\psi^*)(s), \quad s \in S.$$

Under assumptions (A5) and (A9), the set of functions $\{Q_{f^*}^{(m)} \hat{J}_\infty^0(\psi^*)(\cdot)\}$ is uniformly bounded with respect to m and $s \in S$. Taking the limit as $m \rightarrow \infty$ in [\(24\)](#) and making use of [\(22\)](#), we infer that

$$L_{f^*} \hat{J}_\infty^0(\psi^*)(s) = \hat{J}_\infty^0(f^*)(s) = L_{f^*} \hat{J}_\infty^0(f^*)(s), \quad s \in S.$$

Hence, it follows that

$$\beta Q_{f^*} \hat{J}_\infty^0(\psi^*)(s) = \beta Q_{f^*} \hat{J}_\infty^0(f^*)(s), \quad s \in S.$$

Multiplying both sides of this equality by α and adding $u(s, f^*)$, we get

$$u(s, f^*) + \alpha \beta Q_{f^*} \hat{J}_\infty^0(\psi^*)(s) = u(s, f^*) + \alpha \beta Q_{f^*} \hat{J}_\infty^0(f^*)(s) \quad (25)$$

Since $f^*(s) \in A^*(s)$ for each $s \in S$, it follows from [\(25\)](#) that

$$l^*(s) = u(s, f^*) + \alpha \beta Q_{f^*} \hat{J}_\infty^0(f^*)(s), \quad s \in S.$$

Suppose that f^* is not a *SMPE*. Then, there exists some $s \in S$ and $a_s \in A(s)$ such that

$$l^*(s) = u(s, f^*) + \alpha \beta Q_{f^*} \hat{J}_\infty^0(f^*)(s) < u(s, a_s) + \alpha \beta \int_S \hat{J}_\infty^0(f^*)(s') q(ds'|s, a_s).$$

But we know that $\hat{J}_\infty^0(f^*) = L_{f^*} \hat{J}_\infty^0(f^*) = L_{f^*} \hat{J}_\infty^0(\psi^*)$. Thus, using (23) we get

$$\begin{aligned} l^*(s) &< u(s, a_s) + \alpha \beta \int_S L_{f^*} \hat{J}_\infty^0(\psi^*)(s') q(ds'|s, a_s) \\ &= u(s, a_s) + \alpha \beta \int_S L_{\psi^*} \hat{J}_\infty^0(\psi^*)(s') q(ds'|s, a_s) \\ &= u(s, a_s) + \alpha \beta \int_S \hat{J}_\infty^0(\psi^*)(s') q(ds'|s, a_s) \leq l^*(s). \end{aligned}$$

This contradiction completes the proof. $\square$

4.3. Hyperbolic players

The concept of quasi-hyperbolic discounting in dynamic choice models was also inspired by the work of Phelps and Pollak [68]. This idea has been further developed by a number of authors. For example, Harris and Laibson [40] studied a consumption and savings model under uncertainty with “hyperbolic consumers”. Related results were proved by Balbus and Nowak [6] and Nowak [64]. Moreover, quite recently Montiel Olea and Strzalecki [59] provided a natural axiomatisation of quasi-hyperbolic discounting. It is worth emphasising that from mathematical point of view, the equilibrium problem analysed for hyperbolic players is almost the same as in the intergenerational game examined in Subsection 4.1. The only difference is the fact that we need to take into account a countable set of descendants for each generation. Let us now explain the terminology used in Harris and Laibson [40]. We envision an individual decision maker to be a sequence of “selves” indexed by discrete time $t \in T$. The decision maker is then modelled as a distinct player in the sense of non-cooperative game theory. As a result, we deal with a sequential game with countably infinitely many players who each acts only once, but who all care not only about their instantaneous utility in their own period but also about the instantaneous utilities in the consecutive periods, discounted at a given discount rate. Theorem 3 can be obviously formulated in terms of hyperbolic players. A close result is given in Balbus and Nowak [6], where it is assumed that S is an interval of the real line, and transition probability and utility functions satisfy some concavity conditions. Moreover, our proof of Theorem 4 is based upon Theorem 3 and Lemma 1, which allow for consideration of a general state space framework. Therefore, our model includes (as possible applications) a number of examples that deal with a multidimensional state space, e.g., the stochastic production economy with capital and labour exploited in Balbus et al. [7]. As for the work of Harris and Laibson [40], they study a consumption and savings problem with $S = R_+$, and the transition structure described by a certain difference equation with i.i.d. shocks having a distribution with bounded support. Moreover, they assume that the instantaneous utility function u has a relative bounded risk aversion factor. When compared to our condition, we do not put such a limitation. However, we assume that the transition probability function are of certain form expressed in (A5).

5. Invariant distributions

A common issue studied in economic theory and dynamic games concerns the existence of an invariant (or stationary) distribution for the Markov chain induced by the transition function

and an equilibrium strategy, see Futia [32], Stokey et al. [71], Kamihigashi and Stachurski [49], Balbus et al. [7] and references cited therein.

Let $BM(S)$ be the space of all bounded Borel measurable real-valued functions on the Borel space S . By $BC(S)$ we denote the space of all continuous functions in $BM(S)$. Let Q be a transition probability from S to S . Then Q induces the transition operator on $BM(S)$ into itself (denoted for convenience by the same symbol) defined by

$$Qv(s) := \int_S v(s') Q(ds'|s), \quad s \in S, v \in BM(S).$$

The transition operator Q is (strong) Feller, if the mapping $s \mapsto Qv(s)$ is continuous for each $(v \in BM(S)) v \in BC(S)$. A probability measure ν on S is invariant for Q if

$$\nu(B) = \int_S Q(B|s) \nu(ds) \quad \text{for all } B \in \mathcal{B}(S).$$

Let $Q^t(\cdot|s)$ denote the t -step transition probability induced by Q . Assume that S is a complete separable metric space and Q is Feller. If, in addition, there exists some state $s \in S$ such that the family of probability measures $\{Q^t(\cdot|s)\}_{t \in T}$ is tight, then there exists at least one invariant measure for Q , see the Krylov–Bogolioubov theorem, e.g., Chapter 3 in Da Prato and Zabczyk [23].⁴ If we deal with a strong Feller operator Q , then to obtain a unique invariant probability measure for Q , it is sufficient to know that there exists some state $s_0 \in S$ such that for any open neighbourhood B_0 of s_0 , there is some $t \in T$ such that $Q^t(B_0|s) > 0$ for all $s \in S$. Such a state s_0 is called accessible for Q . This result follows from Corollary 2.7 given by Hairer [37]. His proof proceeds along similar lines as the proof of Proposition 4.1.1 in Da Prato and Zabczyk [23].

Let us consider the transition operator Q^* induced by q and a Markov perfect Nash equilibrium $f^* \in F$, that is, $Q^*(\cdot|s) := q(\cdot|s, f^*(s))$, $s \in S$. Assume (A5) and that the finite family of measures $\mu_1, \dots, \mu_l$ is tight, i.e., for any $\epsilon > 0$, there exists a compact set $K \subset S$ such that $\mu_k(K) \geq 1 - \epsilon$ for all $k = 1, \dots, l$ (see Billingsley [17]). If all the functions $s \mapsto g_k(s, f^*(s))$ are continuous, then Q^* is strong Feller. From assumption (A5), it immediately follows that

$$Q^*(K|s) \geq 1 - \epsilon \quad \text{for all } s \in S.$$

This inequality implies that Q^* is tight and thus Q^* has an invariant probability measure ν^* . In order to guarantee the uniqueness of ν^* , it is sufficient to assume that there exists some state $s_0 \in S$ such that for any open neighbourhood B_0 of s_0 , $\mu_k(B_0) > 0$ for all $k = 1, \dots, l$. This, in turn, yields that s_0 is accessible for Q^* . If, on the other hand, we know that for this equilibrium f^* , there exists some k_0 such that $g_{k_0}(s, f^*(s)) > 0$ for all $s \in S$ and $\mu_{k_0}(B_0) > 0$ for any open neighbourhood B_0 of s_0 , then s_0 is accessible and we get again the uniqueness of ν^* . The above remarks apply to the cases studied in Section 6 (or in Balbus et al. [7]), for which the functions f^* and g_k , $k = 1, \dots, l$, are continuous. However, Lemma 1 does not ensure the strong Feller property for Q^* because f^* may be discontinuous. Nevertheless, the process, governed by Q^* , may still possess a unique invariant probability measure ν^* . Indeed, if assumption (A5) is satisfied and there is j_0 such that $g_{j_0}(s, a) \geq \epsilon$ for some $\epsilon > 0$ and for all $(s, a) \in C$, then

$$Q^*(B|s) = q(B|s, f^*(s)) \geq \epsilon \mu_{j_0}(B) \quad \text{for all } B \in \mathcal{B}(S), s \in S. \quad (26)$$

⁴ We thank Martin Hairer and John Stachurski for drawing our attention to this result.

This, in turn, implies the strong geometric ergodicity of the Markov process induced by Q^* . More precisely, (26) guarantees that t -step transition probabilities $(Q^*)^t$ converge to the unique invariant distribution at geometric rate as $t \rightarrow \infty$, see Lemma 3.3 in Hernández-Lerma [44].

6. Special cases with transitions allowing for atoms

In this section, we study a certain class of intergenerational strategic bequests models with one-dimensional state space and concave utility functions. Our assumptions imposed on the transition probability and utility functions resemble the ones accepted in Amir [3,4], Nowak [62,64], Balbus et al. [7]. However, the novelty of our approach is based upon the fact that we deduce the existence of a SMPE from Theorem 1. This is only possible thanks to specific concavity conditions used in these models. In addition, it is also worthy to stress that the results on stationary Markov perfect equilibria in the bequest games with aggregator function U^r , $r < 0$, are new.

Let $S = [0, \bar{s}] \subset R_+$ or $S = R_+$. The set S is referred to as the set of *renewable resources*. For each $s \in S$, $A(s) := [0, s]$ is the set of resources available for consumption in state $s \in S$. It is obvious that $C = \{(s, a) : s \in S, a \in A(s)\}$ is a complete lattice in R^2 with component-wise order. Let us assume that u be a utility function of one variable (consumption) $a \in A(s)$, for any $s \in S$. To be more precise, we should write that $u(s, a) = w(a)$ for each $(s, a) \in C$ and some function w of one variable. But for convenience, we use the same symbol u . A similar convention is applied to the functions $v_1, \dots, v_m$. A pure Markov strategy $f \in F$ specifies a consumption level $f(s) \in A(s)$ for each $s \in S$.

We make the following additional assumptions on the primitive data.

(C1) The transition probability q is of the form

$$q(B|s, a) = g(s - a)\mu_1(B) + (1 - g(s - a))\mu_2(B), \quad B \in \mathcal{B}(S), (s, a) \in C, \quad (27)$$

where $g : S \mapsto [0, 1]$ is a continuous, increasing, concave function and μ_1, μ_2 are probability measures on S such that $\mu_1 \succ \mu_2$, i.e., μ_1 (first order) stochastically dominates μ_2 .

We have $\mu = \frac{(\mu_1 + \mu_2)}{2}$.

(C2) The functions $u, v_1, \dots, v_m$ are non-negative, continuous and increasing on R_+ .

(C3) The function u is strictly concave.

(C4) For $k = 1, 2$ and $j = 1, \dots, m$, the integrals $\int_S v_j(y)\mu_k(dy)$ are finite.

Remark 2. Recall that $\mu_1 \succ \mu_2$ if and only if for an increasing function v , we get that $\int_S v(s)\mu_1(ds) > \int_S v(s)\mu_2(ds)$.

Remark 3. Similar assumptions to (C1) have also been used in the analysis of n -person Markov game models that possess a wide range of applications to economics; see Amir [4], Curtat [22], Horst [46] and Nowak [63] for general accounts.

We can now state our result for $r = 0$.

Proposition 1. Under assumptions (C1)–(C4) there exists a SMPE $f^* \in F$. Moreover, f^* is non-decreasing and Lipschitz continuous with constant one.

Proof. By Theorem 1 there exists a *SMPE* $\psi^* \in \Psi$. Consider the function

$$p_m^0(s, \psi^*)(a) := u(a) + \int_S J_m^0(\psi^*)(s') q(ds'|s, a), \quad a \in A(s) = [0, s].$$

Under condition (C1), we have

$$p_m^0(s, \psi^*)(a) = u(a) + \int_S J_m^0(\psi^*)(s') \mu_2(ds') + D(\psi^*)g(s-a), \quad (28)$$

where

$$D(\psi^*) = \int_S J_m^0(\psi^*)(s') \mu_1(ds') - \int_S J_m^0(\psi^*)(s') \mu_2(ds').$$

We show that $D(\psi^*) > 0$. Indeed, on the contrary, assume that $D(\psi^*) \leq 0$. By (28), the function $p_m^0(s, \psi^*)(\cdot)$ is increasing on $A(s)$ for any $s > 0$. Therefore, we have

$$\arg \max_{a \in A(s)} p_m^0(s, \psi^*)(a) = s \quad \text{for each } s \in S.$$

Since ψ^* is an equilibrium, we must have $\psi^*(s) = s$ for all $s \in S$. Moreover, $D(\psi^*) = D(f_0)$ where $f_0(s) := s$ for $s \in S$ and

$$D(f_0) = \int_S J^0(f_0)(s') \mu_1(ds') - \int_S J^0(f_0)(s') \mu_2(ds').$$

Observe that, since v_1 is increasing and $f_0(s') = s'$ for all $s' \in S$, we have

$$J_m^0(f_0)(s') = v_1(s') + C_0,$$

where

$$C_0 = \sum_{k=2}^m \int_S v_k(y) \mu_0(dy) \quad \text{and} \quad \mu_0 = g(0)\mu_1 + (1 - g(0))\mu_2.$$

Hence, it follows that $J_m^0(f_0)$ is increasing on S . Since $\mu_1 > \mu_2$, we have $D(\psi^*) = D(f_0) > 0$, which leads to a contradiction. Therefore, we must have $D(\psi^*) > 0$ and the function $p_m^0(s, \psi^*)(\cdot)$ defined in (28) is strictly concave on $A(s)$ for each $s > 0$. Thus, $f^*(s) := \arg \max_{a \in A(s)} p_m^0(s, \psi^*)(a)$ is uniquely determined for any $s \in S$. (Observe that $f^*(0) = 0$.) Hence, it follows that $\psi^*(s) = f^*(s)$ for all $s \in S$, so f^* is a *SMPE*. Furthermore, note that

$$(s, a) \mapsto u(a) + \int_S J_m^0(f^*)(s') \mu_2(ds') + D(f^*)g(s-a)$$

with $D(f^*) = D(\psi^*) > 0$ is supermodular on the lattice C (see Lemma 0.2 in Amir [3]). Therefore, by the well-known theorem of Topkis [73], we infer that f^* is non-decreasing and Lipschitz with constant one. $\square$

We now apply Theorem 1 to a model with $r < 0$. In this case, we make an additional assumption on the function g .

(C5) The function g in (27) is of the form

$$g(s-a) := 1 - e^{-b(s-a)}, \quad (s, a) \in C,$$

with some $b > 0$.

Proposition 2. Under assumptions (C1)–(C5) there exists a SMPE $f^* \in F$. Moreover, f^* is continuous.

Proof. By Theorem 1, there exists a randomised SMPE $\psi^* \in \Psi$. Consider the function

$$p_m^r(s, \psi^*)(a) = -e^{ru(a)} \int_S J_m^r(\psi^*)(s') q(ds'|s, a), \quad a \in A(s) = [0, s],$$

where $J_m^r(\psi^*)$ is defined in (13). We have

$$p_m^r(s, \psi^*)(a) = -e^{ru(a)} \left(\int_S J_m^r(\psi^*)(s') \mu_2(ds') + D^r(\psi^*)g(s-a) \right)$$

with

$$D^r(\psi^*) = \int_S J_m^r(\psi^*)(s') \mu_1(ds') - \int_S J_m^r(\psi^*)(s') \mu_2(ds').$$

Suppose that $D^r(\psi^*) \geq 0$. Then the function

$$\eta(a) := \int_S J_m^r(\psi^*)(s') \mu_2(ds') + D^r(\psi^*)g(s-a)$$

is non-increasing. Moreover, we know that $\eta(a) > 0$ for all $a \in A(s)$. Thus, $a \mapsto e^{ru(a)}\eta(a)$ is decreasing in $a \in A(s)$ and attains its minimum at the point $a := s \in A(s)$. Hence, it follows that

$$\arg \max_{a \in A(s)} p_m^r(s, \psi^*)(a) = s \quad \text{for each } s \in S$$

and, since ψ^* is a SMPE, we must have $\psi^*(s) = f_0(s) = s$ for all $s \in S$. But, then

$$J_m^r(\psi^*)(s') = J_m^r(f_0)(s') = e^{rv_1(s')} C_r, \quad \text{with } C_r = \prod_{k=2}^m \int_S e^{rv_k(y)} \mu_1(dy) > 0.$$

Hence $J_m^r(\psi^*)(\cdot) = J_m^r(f_0)(\cdot)$ is decreasing on S . Since $\mu_1 \succ \mu_2$, we get that $D^r(\psi^*) < 0$, which leads to a contradiction with our assumption. Thus, we must have $D^r(\psi^*) < 0$. Moreover, observe that

$$\begin{aligned} p_m^r(s, \psi^*)(a) &= -e^{ru(a)} \left(\int_S J_m^r(\psi^*)(s') \mu_1(ds') - D^r(\psi^*)e^{b(a-s)} \right) \\ &= -e^{ru(a)} \int_S J_m^r(\psi^*)(s') \mu_1(ds') + D^r(\psi^*)e^{ru(a)+b(a-s)}. \end{aligned} \quad (29)$$

Since $D^r(\psi^*) < 0$, the function $p_m^r(s, \psi^*)(\cdot)$ is strictly concave on $A(s)$ for each $s \in S$. Therefore, $f^*(s) := \arg \max_{a \in A(s)} p_m^r(s, \psi^*)(a)$ is uniquely determined for any $s \in S$. Hence, it follows that $\psi^*(s) = f^*(s)$ for all $s \in S$, so f^* is a SMPE. Clearly, f^* is continuous. $\square$

Remark 4. If we only assume (C1), then the function $e^{ru(a)}(1 - g(s - a))$ (that is multiplied by $D^r(\psi^*)$ in (29)) may not be concave. In this case, the best reply function f^* considered in the proof of Proposition 2 may not be unique and we cannot claim that $\psi^* = f^*$.

Remark 5. In Proposition 2 our condition (C5) guarantees that the function $a \mapsto p^r(s, \psi^*)(a)$ is strictly concave, and therefore there is only one point at which the maximum is attained. However, it is well-known that quasi-concavity is a considerable generalisation of concavity, see Chapter 3.4 in Boyd and Vandenberg [18]. Let $h : R \mapsto R$ be a continuous function with a convex domain. Recall that a function h is quasi-concave, if the set $\{x \in R : h(x) \geq d\}$ is convex for any constant d or h is quasi-concave if and only if at least one of the following conditions holds:

- (a) h is non-decreasing,
- (b) h is non-increasing,
- (c) there is a point x_0 in the domain of h such that for $x < x_0$ (and x is in the domain of h), h is non-decreasing and for $x > x_0$ (and x is in the domain of h) is non-increasing.

The point x_0 can be chosen as any point, which is a global maximiser of h . In other words, the quasi-concave function h is unimodal. Condition (c) for a twice differentiable h reduces to the simple requirement $h'(x) = 0 \implies h''(x) \leq 0$. Obviously, if we additionally expect to have a unique maximiser of h , then h must be either increasing or decreasing or there must be a unique point x^* such that

$$h'(x^*) = 0 \implies h''(x^*) < 0. \quad (30)$$

Let us now turn to our objective function

$$p_m^r(s, \psi^*)(a) = -e^{ru(a)}(G_2 + (G_1 - G_2)g(s - a)),$$

where ψ^* is a SMPE, $r < 0$, and

$$G_i = \int_S J_m^r(\psi^*)(s') \mu_i(ds'), \quad i = 1, 2.$$

Note that from the proof of Proposition 2, we have that $G_1 < G_2$. Assume further that $p_m^r(s, \psi^*)(\cdot)$ is twice differentiable on S . Condition (30) can be read as follows: for fixed $s \in S \setminus \{0\}$

$$\begin{aligned} & \left. \frac{dp_m^r(s, \psi^*)(a)}{da} \right|_{a=a^*} \\ &= -e^{ru(a^*)}((G_1 - G_2)(ru'(a^*)g(s - a^*) - g'(s - a^*)) + ru'(a^*)G_2) = 0 \\ \implies & \frac{u''(a^*)}{u'(a^*)}g'(s - a^*) - ru'(a^*)g'(s - a^*) + g''(s - a^*) < 0. \end{aligned} \quad (31)$$

Observe that the left-hand side in the last display is not always negative and therefore, this condition is essential. Summing up, we come to the following conclusion. If the $a \mapsto p_m^r(s, \psi^*)(a)$ defined on $A(s)$ is either increasing or decreasing or (31) holds, then there exists a SMPE $f^* \in F$, and moreover, f^* is continuous.

Example 4. Assume that $S = [0, 1]$, $u(a) = \sqrt{a}/4$, $g(y) = y$ for $y = s - a$, and $r = -1$. Moreover, let μ_1 , μ_2 and $v_1, \dots, v_m$ be chosen in such a way that $G_1 = \frac{1}{2}$ and $G_2 = \frac{3}{4}$. Then, our objective function equals

$$p_m^r(s, \psi^*)(a) = -e^{-\sqrt{a}/4} \left(\frac{3}{4} - \frac{1}{4}(s - a) \right), \quad a \in [0, s],$$

and its first derivative is

$$\frac{dp_m^r(s, \psi^*)(a)}{da} = \frac{1}{32\sqrt{a}} e^{\sqrt{a}/4} (a - 8\sqrt{a} + 3 - s), \quad a \in [0, s].$$

Note that $a \mapsto (a - 8\sqrt{a} + 3 - s)$ is increasing for $s \in [0, \frac{9}{64}]$, hence the maximum is at the point s . If, on the other hand, $s \in (\frac{9}{64}, 1]$, then $a \mapsto (a - 8\sqrt{a} + 3 - s)$ attains the maximum at a certain point $a^* \in (0, s)$. Clearly, condition (31) is satisfied for a^* , since for each $s \in (9/64, 1]$ we have that

$$\frac{u''(a^*)}{u'(a^*)} g'(s - a^*) - r u'(a^*) g'(s - a^*) + g''(s - a^*) = -\frac{1}{2a^*} + \frac{1}{8\sqrt{a^*}} < 0.$$

Therefore, the best reply function for ψ^* is uniquely determined and equals

$$f^*(s) = \begin{cases} s, & \text{for } s \in [0, \frac{9}{64}] \\ 29 + s - 8\sqrt{13 + s}, & \text{for } s \in (\frac{9}{64}, 1]. \end{cases}$$

Hence, $f^* = \psi^*$ is a *SMPE*. It is worth pointing out that although the function $s \mapsto f^*(s)$ is decreasing on $(\frac{9}{64}, 1]$, the function $R^r(s, f^*, f^*)$ is increasing on $[0, 1]$. Moreover, if $s \in [0.9309, 1]$ the function $p^r(s, \psi^*)(\cdot)$ is not concave on $A(s)$, it has inflection point in $A(s)$.

We close this section with a result on intergenerational games involving countably many descendants.

Proposition 3. Assume that u is bounded, continuous, increasing and non-negative on R_+ . Suppose that (C1) and (C3)–(C5) hold and $r < 0$. Then, the intergenerational stochastic game involving countably many descendants has a *SMPE*.

Proof. By Theorem 3, the game has a randomised *SMPE* $\psi^* \in \Psi$. The remaining part of the proof is the same as that of Proposition 2 with $J_m^r(\psi^*)$ replaced by $J_\infty^r(\psi^*)$. $\square$

7. Concluding remarks

This paper is concerned with a pretty general model of intergenerational stochastic game with a Borel state space and atomless additive transition probabilities. The obtained results may have applications to analysing various specific bequest games or OLG models with multidimensional state spaces.⁵ The key idea in our approach to get the existence of *SMPE* is to utilise the purification theorem due to Dvoretzky, Wald and Wolfowitz [27]. We emphasise that this theorem

⁵ The need of examination of multidimensional case was already postulated by Bernheim and Ray [14]. To the best of our knowledge Balbus et al. [7,9] are the only papers that study related models to ours and deal with a two-dimensional state space.

has not been applied (in general) to stochastic games. The advantage of using this tool is the fact that it enables us to impose relatively weak conditions on the primitive data both in games with finitely many descendants and also in certain games with infinitely many descendants and a recursive structure. The latter point means that our approach also works in some consumption-savings models with hyperbolic players. Up to now, intergenerational stochastic games have been studied with a linear aggregator function, which corresponds to the case $r = 0$. Our novel idea, in this paper, is to introduce a new way of aggregating partial utilities, namely with the aid of the exponential function with some negative coefficient r reflecting an additional attitude to risk. This approach is equivalent to looking at the certainty equivalent of the sum of random partial utilities received along trajectories of the game process. This certainty equivalent is known in finance as the entropic risk measure. For its applications in various areas of research the reader is referred to Subsection 3.1 and references cited therein. Although this manner of aggregation was already used in dynamic games, there is no contribution of such an approach to intergenerational stochastic games (or more generally OLG models). Our toy examples in Subsection 3.2 aim at explaining the difference between the two cases $r = 0$ and $r < 0$ and providing the meaning of being a risk averse generation in a bequest stochastic game.

Our paper does not answer all inspiring questions that stem from proposed approach. Some of them are left open. First, we do not examine the uniqueness issue for *SMPE*. This problem was partially resolved for the linear aggregation function (i.e., for $r = 0$ according to our terminology) by Balbus et al. [7,8]. The methods used in the aforementioned papers strongly exploit some geometrical properties of monotone mappings and are based on Theorem 3.2.5 in Guo et al. [36]. However, in order to apply the result from Guo et al. [36] the authors have to assume that the state space is an interval in the real line and one of the measures involved in the transition probability is the Dirac measure concentrated at zero. These conditions entail some concavity properties of the functions used in the model. In our case, when $r < 0$, the use of Theorem 3.2.5 in Guo et al. [36] is not helpful. As Example 1 in Subsection 3.2 illustrates, the functions involved in our analysis are neither convex nor concave and the argmax correspondence may have non-convex values. Uniqueness is a significant issue, but it must be studied in different way that suggested so far in the literature. Nonetheless, at the moment we are not aware of any techniques that can be useful in solving this problem. The second crucial question is a computation of *SMPE*. For the case $r = 0$, this problem was settled in Balbus et al. [7–9] for straightforward games by means of globally stable iterative procedures. However, these algorithms are available thanks to the particular form of transition law that assumes that one of the measures is the Dirac measure concentrated at zero. As our examples show, the case $r < 0$ leads to some complicated nonlinear equations, whereas a solution for the case $r = 0$ is pretty easy to find. *SMPE* obtained in Examples 1 and 2 exhibit “extreme behaviour”, i.e., to consume everything or nothing. This is due to the fact that the functions u and v are linear. More stimulating equilibria should be “interior”. However, calculating an “interior” *SMPE* is a challenging problem, even if u and v are power functions. In this setup, the approach based on monotone value function operator methods does not lead to a solution. Hence again, there is a need of development of new techniques. We hope that this paper will inspire researchers to the study of computational issues for *SMPE* in risk sensitive models.

Acknowledgments

We thank two anonymous referees and the editor for helpful reviews and clarifying some important points that improve the presentation of the paper.

Appendix A

Let $L_1(\mu) := L_1(S, \mu)$ ($L_\infty(\mu) := L_\infty(S, \mu)$) denote the Banach space of all μ -integrable (μ -essentially bounded) real-valued measurable functions on S . Endow $L_1(\mu)$ with the weak topology $\sigma(L_1(\mu), L_\infty(\mu))$. The weak convergence of a sequence $\{h_n\}$ to some $h \in L_1(\mu)$ is denoted by: $h_n \xrightarrow{\omega} h$ in $L_1(\mu)$ and it basically says that $\int_B h_n(x) \mu(dx) \rightarrow \int_B h(x) \mu(dx)$ as $n \rightarrow \infty$, for every $B \in \mathcal{B}(S)$. The proof of existence of an equilibrium in the sense of Definition 1 rests upon a fixed point argument. We endow the strategy space with a compact topology. Let Ψ^μ denote the quotient space of all equivalence classes of functions $\psi \in \Psi$ which are equal μ -a.e. where $\mu = (\mu_1 + \dots + \mu_l)/l$. Since the set $A(s)$ is compact, Ψ^μ is compact and metrisable when endowed with the weak-star topology. For the details we refer the reader to Balder [10] or Chapter IV in Warga [74]. Here, we only mention that a sequence $\{\psi_n\}$ converges to ψ in Ψ^μ if and only if for every $w : C \rightarrow R$ such that $w(s, \cdot)$ is continuous on $A(s)$ for each $s \in S$, $w(\cdot, a)$ is measurable for each $a \in A(s)$, and $s \mapsto \max_{a \in A(s)} |w(s, a)|$ is μ -integrable over S (i.e., w is a Carathéodory function), we have

$$\int_S \int_{A(s)} w(s, a) \psi_n(da|s) \mu(ds) \rightarrow \int_S \int_{A(s)} w(s, a) \psi(da|s) \mu(ds) \quad \text{as } n \rightarrow \infty.$$

Replacing w by $w 1_B$ above, where 1_B is the indicator function of any set $B \in \mathcal{B}(S)$, we can write that

$$\int_B w(x, \psi_n) \mu(dx) \rightarrow \int_B w(x, \psi) \mu(dx) \quad \text{as } n \rightarrow \infty.$$

In other words, the sequence $\{w(\cdot, \psi_n)\}$ weakly converges to $w(\cdot, \psi)$ in $L_1(\mu)$. Let $\tilde{w}$ be a Carathéodory function. Then under assumption (A5), we have

$$\sup_{\psi \in \Psi} \sup_{(s, a) \in C} \left| \int_S \tilde{w}(s', \psi) q(ds'|s, a) \right| \leq \max_{1 \leq k \leq l} \int_S \max_{a \in A(s')} |\tilde{w}(s', a)| \mu_k(ds') < \infty. \quad (32)$$

We now define a special class Γ of pairs $(\{h_n\}, h)$ where $h \in L_1(\mu)$ and $\{h_n\}$ is a sequence in $L_1(\mu)$. Namely, $(\{h_n\}, h) \in \Gamma$ if $h_n \xrightarrow{\omega} h$ in $L_1(\mu)$ and, for any $k = 1, \dots, l$, $\int_S h_n(x) \mu_k(dx) \rightarrow \int_S h(x) \mu_k(dx)$ as $n \rightarrow \infty$.

Lemma 2. Assume that (A5) holds and $(\{h_n\}, h) \in \Gamma$. Let w be a Carathéodory function. Define

$$h'_n(s) := \int_{A(s)} \int_S h_n(s') q(ds'|s, a) w(s, a) \psi_n(da|s)$$

and

$$h'(s) := \int_{A(s)} \int_S h(s') q(ds'|s, a) w(s, a) \psi(da|s), \quad s \in S.$$

Then $(\{h'_n\}, h') \in \Gamma$.

Proof. Note that

$$h'_n(s) - h'(s) = \int_{A(s)} \int_S (h_n(s') - h(s')) q(ds'|s, a) w(s, a) \psi_n(da|s) + Z_n(s), \quad (33)$$

where

$$Z_n(s) := \left(\int_{A(s)} \int_S h(s') q(ds'|s, a) w(s, a) \psi_n(da|s) - \int_{A(s)} \int_S h(s') q(ds'|s, a) w(s, a) \psi(da|s) \right).$$

We notice that if $h \in L_1(\mu)$, then by assumption (A5), it follows that the function defined as $(s, a) \mapsto \int_S h(y) q(dy|s, a) w(s, a)$ is Carathéodory. Thus, Z_n tends to zero weakly in $L_1(\mu)$ by the definition of $\psi_n \rightarrow \psi$ in the space Ψ^μ . Denote by ρ_k a density of μ_k with respect to μ . Clearly, $(s, a) \mapsto \int_S h(y) q(dy|s, a) w(s, a) \rho_k(s)$ is also a Carathéodory function. Thus, the convergence $\psi_n \rightarrow \psi$ in Ψ^μ implies that

$$\begin{aligned} & \int_S \int_{A(s)} \int_S h(s') q(ds'|s, a) w(s, a) \psi_n(da|s) \mu_k(ds) \\ &= \int_S \int_{A(s)} \int_S h(s') q(ds'|s, a) w(s, a) \rho_k(s) \psi_n(da|s) \mu(ds) \\ &\rightarrow \int_S \int_{A(s)} \int_S h(s') q(ds'|s, a) w(s, a) \rho_k(s) \psi(da|s) \mu(ds) \\ &= \int_S \int_{A(s)} \int_S h(s') q(ds'|s, a) w(s, a) \psi(da|s) \mu_k(ds) \end{aligned} \quad (34)$$

as $n \rightarrow \infty$, $k = 1, \dots, l$.

Let us now observe that since $\int_S h_n(s') \mu_k(ds') \rightarrow \int_S h(s') \mu_k(ds')$ as $n \rightarrow \infty$, for any $k = 1, \dots, l$, it follows that there exists some $\eta > 0$ such that

$$\max_{1 \leq k \leq l} \left| \int_S (h_n(s') - h(s')) \mu_k(ds') \right| \leq \eta.$$

For the first term in (33), we have that

$$\begin{aligned} & \left| \int_{A(s)} \int_S (h_n(s') - h(s')) q(ds'|s, a) w(s, a) \psi_n(da|s) \right| \\ & \leq \max_{a \in A(s)} |w(s, a)| \max_{1 \leq k \leq l} \left| \int_S (h_n(s') - h(s')) \mu_k(ds') \right| \leq \eta \max_{a \in A(s)} |w(s, a)|. \end{aligned} \quad (35)$$

From (35), it follows that

$$Y_n(s) := \int_{A(s)} \int_S (h_n(s') - h(s')) q(ds'|s, a) w(s, a) \psi_n(da|s) \rightarrow 0$$

as $n \rightarrow \infty$. For each $k = 1, \dots, l$, we have $|Y_n(s) \rho_k(s)| \leq \eta \max_{a \in A(s)} |w(s, a)| \rho_k(s)$ for all $n \geq 1$. Since $\int_S \max_{a \in A(s)} |w(s, a)| \mu_k(ds) < \infty$, then by the Lebesgue dominated convergence theorem, we conclude that

$$\int_B Y_n(s') \mu(ds') \rightarrow 0 \quad \text{and} \quad \int_S Y_n(s') \mu_k(ds') = \int_S Y_n(s') \rho_k(s') \mu(ds') \rightarrow 0 \quad (36)$$

as $n \rightarrow \infty$ for any $B \in \mathcal{B}(S)$. From (34), the fact that $Z_n \xrightarrow{\omega} 0$ in $L_1(\mu)$ and (36), it follows that $(\{h'_n\}, h') \in \Gamma$.

Lemma 3. Assume that (A1)–(A7) are satisfied. If $\psi_n \rightarrow \psi$ in Ψ^μ , then $(\{J_m^r(\psi_n)\}, J_m^r(\psi)) \in \Gamma$.

Proof. Let us first consider $r < 0$. Assume that $\psi_n \rightarrow \psi$ in Ψ^μ as $n \rightarrow \infty$. Put $\tilde{V}_m(\psi_n)(y) := V_m(y, \psi_n)$ and $\tilde{V}_m(\psi)(y) := V_m(y, \psi)$, $y \in S$. From our assumptions (A5)–(A7), it follows that $(\{\tilde{V}_m(\psi_n)\}, \tilde{V}_m(\psi)) \in \Gamma$. Further, observe that by (32) with $\tilde{w} = V_m$, Lemma 2 with $w = V_{m-1}$, we deduce that

$$(\{(V_{m-1}Q)_{\psi_n} \tilde{V}_m(\psi_n)\}, (V_{m-1}Q)_\psi \tilde{V}_m(\psi)) \in \Gamma.$$

Continuing this procedure and applying (32) and Lemma 2 ($m-1$) times, we infer that

$$(\{(V_1Q)_{\psi_n} \cdots (V_{m-1}Q)_{\psi_n} \tilde{V}_m(\psi_n)\}, (V_1Q)_\psi \cdots (V_{m-1}Q)_\psi \tilde{V}_m(\psi)) \in \Gamma.$$

This fact and (13) imply that

$$(\{J_m^r(\psi_n)\}, J_m^r(\psi)) \in \Gamma.$$

Let us now assume that $r = 0$. Put $\tilde{v}_j(\psi_n)(y) := v_j(y, \psi_n)$ and $\tilde{v}_j(\psi)(y) := v_j(y, \psi)$, $y \in S$. Then, we have $(\{\tilde{v}_j(\psi_n)\}, \tilde{v}_j(\psi)) \in \Gamma$ for $j = 1, \dots, m$. From (32) with $\tilde{w} = v_j$, Lemma 2 with $w \equiv 1$, it follows that for $j = 2, \dots, m$, we have $(\{Q_{\psi_n} \tilde{v}_j(\psi_n)\}, Q_\psi \tilde{v}_j(\psi)) \in \Gamma$. Applying (32) and Lemma 2 again for $j > 2$ ($(j-2)$ times) with $w \equiv 1$, we finally get

$$(\{Q_{\psi_n}^{(j-1)} \tilde{v}_j(\psi_n)\}, Q_\psi^{(j-1)} \tilde{v}_j(\psi)) \in \Gamma.$$

Thus,

$$\left(\left\{ \tilde{v}_j(\psi_n) + \sum_{j=2}^m Q_{\psi_n}^{(j-1)} \tilde{v}_j(\psi_n) \right\}, \tilde{v}_j(\psi) + \sum_{j=2}^m Q_\psi^{(j-1)} \tilde{v}_j(\psi) \right) \in \Gamma.$$

This fact and (14) yield that $(\{J_m^0(\psi_n)\}, J_m^0(\psi)) \in \Gamma$.

Lemma 4. Assume (A1)–(A7) and that $\psi_n \rightarrow \psi$ in Ψ^μ . Then

$$\sup_{\phi \in \Psi} R_m^r(s, \phi, \psi_n) \rightarrow \sup_{\phi \in \Psi} R_m^r(s, \phi, \psi)$$

as $n \rightarrow \infty$.

Proof. Let $r < 0$ and observe that

$$\begin{aligned} & \left| \sup_{\phi \in \Psi} R_m^r(s, \phi, \psi_n) - \sup_{\phi \in \Psi} R_m^r(s, \phi, \psi) \right| \\ & \leq \max_{a \in A(s)} \left[e^{ru(s,a)} \sup_{(s,a) \in C} \left| \int_S (J_m^r(\psi_n)(s') - J_m^r(\psi)(s')) q(ds'|s, a) \right| \right]. \end{aligned}$$

For $r = 0$, we have that

$$\begin{aligned} & \left| \sup_{\phi \in \Psi} R_m^0(s, \phi, \psi_n) - \sup_{\phi \in \Psi} R_m^0(s, \phi, \psi) \right| \\ & \leq \sup_{(s,a) \in C} \left| \int_S (J_m^0(\psi_n)(s') - J_m^0(\psi)(s')) q(ds'|s, a) \right|. \end{aligned}$$

Now, the result easily follows from assumptions (A2), (A6), (A5) and Lemma 3.

Proof of Theorem 1. Let $\psi \in \Psi^\mu$. Under our assumptions the set $\tilde{C} := \{(s, \nu) : s \in S, \nu \in P(A(s))\}$ is Borel and $P(A(s))$ is compact. Define the set

$$F(\psi)(s) := \left\{ \eta \in P(A(s)) : R_m^r(s, \eta, \psi) = \sup_{\nu \in P(A(s))} R_m^r(s, \nu, \psi) \right\}$$

for each $s \in S$. From Nowak and Raghavan [65] (see p. 523), the set-valued mapping $s \mapsto F(\psi)(s)$ admits a Borel measurable selector. Let $G(\psi) \subset \Psi^\mu$ denote the set of all μ -equivalence classes of Borel measurable selectors of $s \mapsto F(\psi)(s)$. We shall prove that $\psi \mapsto G(\psi)$ is upper semicontinuous. Let $\psi_n \rightarrow \psi \in \Psi^\mu$. Assume that $\phi_n \in G(\psi_n)$ for each n and $\phi_n \rightarrow \phi$ in Ψ^μ as $n \rightarrow \infty$. We have that

$$R_m^r(s, \phi_n, \psi_n) = \max_{\varphi \in \Psi} R_m^r(s, \varphi, \psi_n) = \max_{\nu \in P(A(s))} R_m^r(s, \nu, \psi_n), \quad \mu\text{-a.e.}$$

Using Lemmas 2 and 3, it easily follows that $R_m^r(\cdot, \phi_n, \psi_n) \xrightarrow{\omega} R_m^r(\cdot, \phi, \psi)$ in $L_1(\mu)$. On the other hand, by Lemma 4,

$$\max_{\varphi \in \Psi} R_m^r(s, \varphi, \psi_n) \rightarrow \max_{\varphi \in \Psi} R_m^r(s, \varphi, \psi)$$

for every $s \in S$. Thus, we have

$$R_m^r(s, \phi, \psi) = \max_{\varphi \in \Psi} R_m^r(s, \varphi, \psi), \quad \mu\text{-a.e.}$$

In other words, $\phi \in G(\psi)$ which completes the proof of the upper semicontinuity of G . By Glicksberg fixed point theorem [34], there exists some $\phi^* \in \Psi^\mu$ such that $\phi^* \in G(\phi^*)$. Hence, there is a measurable set $S_1 \subset S$ such that $\mu(S_1) = 1$ and

$$R_m^r(s, \phi^*, \phi^*) = \max_{\nu \in P(A(s))} R_m^r(s, \nu, \phi^*), \quad s \in S_1. \quad (37)$$

By Corollary 1 in Brown and Purves [19] there exists a Borel measurable selector φ^* from $S \setminus S_1$ into A such that $\varphi^*(s) \in A(s)$ and

$$R_m^r(s, \varphi^*, \phi^*) = \max_{\nu \in P(A(s))} R_m^r(s, \nu, \phi^*) \quad (38)$$

for each $s \in S \setminus S_1$. Define $\psi^*(s) = \phi^*(s)$ for $s \in S_1$ and $\psi^*(s) = \varphi^*(s)$ for $s \in S \setminus S_1$. Since $q(\cdot|s, a) \ll \mu$ for all $(s, a) \in C$, we have

$$R_m^r(s, \psi^*, \phi^*) = R_m^r(s, \psi^*, \psi^*)$$

for all $s \in S$. This fact, (37) and (38) imply that ψ^* is a SMPE.

Proof of Theorem 2. Let $\psi^* \in \Psi$ be a *SMPE* and assume first that $r < 0$. Recall that $\tilde{V}_m(\psi^*)(y) = V_m(y, \psi^*)$ for $y \in S$. By making use of (12), we define the functions:

$$\begin{aligned} w_m(s, a) &:= V_m(s, a) \\ w_{m-1}(s, a) &:= V_{m-1}(s, a) \int_S w_m(s', \psi^*) q(ds' | s, a) \\ &= V_{m-1}(s, a) \int_S \tilde{V}_m(\psi^*)(s') q(ds' | s, a), \\ w_{m-2}(s, a) &:= V_{m-2}(s, a) \int_S w_{m-1}(s', \psi^*) q(ds' | s, a) \\ &= V_{m-2}(s, a) \int_S (V_{m-1} Q)_{\psi^*} \tilde{V}_m(\psi^*)(s') q(ds' | s, a), \\ &\vdots \\ w_1(s, a) &:= V_1(s, a) \int_S w_2(s', \psi^*) q(ds' | s, a) \\ &= V_1(s, a) \int_S (V_2 Q)_{\psi^*} \cdots (V_{m-1} Q)_{\psi^*} \tilde{V}_m(\psi^*)(s') q(ds' | s, a). \end{aligned}$$

Obviously, by (13), it follows that $w_1(s, \psi^*) = J_m^r(\psi^*)(s)$, $s \in S$. Next we put

$$A^*(s) = \arg \max_{a \in A(s)} p_m^r(s, \psi^*)(a),$$

where $p_m^r(s, \psi^*)(\cdot)$ is defined in (16). By Corollary 1 in Brown and Purves [19], it is easily seen that $A^*(s) \neq \emptyset$. Moreover, $\psi^*(A^*(s) | s) = 1$ for each $s \in S$. From Lemma 1, we infer that there exists a measurable mapping $f^* \in F$ from S into A such that $f^*(s) \in A^*(s)$ for each $s \in S$, and

$$\begin{aligned} \int_S \int_{A(s')} w_j(s', a') \psi^*(da' | s') \mu_k(ds') &= \int_S \int_{A^*(s)} w_j(s', a') \psi^*(da' | s') \mu_k(ds') \\ &= \int_S w_j(s', f^*(s')) \mu_k(ds'), \end{aligned}$$

for $j = 1, \dots, m$ and $k = 1, \dots, l$. This fact and assumption (A5) imply that

$$\int_S \int_{A(s')} w_j(s', a') \psi^*(da' | s') q(ds' | s, a) = \int_S w_j(s', f^*(s')) q(ds' | s, a) \quad (39)$$

for all $(s, a) \in C$ and $j = 1, \dots, m$. Now we claim that $f^* \in F$ is a *SMPE*. Indeed, applying (39) to function $w_m, w_{m-1}, \dots, w_1$ in turn, we finally arrive at the following equality

$$\begin{aligned} \int_S w_1(s', \psi^*) q(ds' | s, a) &= \int_S (V_1 Q)_{f^*} (V_2 Q)_{f^*} \cdots (V_{m-1} Q)_{f^*} \tilde{V}_m(f^*)(s') q(ds' | s, a), \\ (s, a) &\in C. \end{aligned}$$

Therefore,

$$\begin{aligned} p_m^r(s, \psi^*)(a) &= -e^{ru(s,a)} \int_S w_1(s', \psi^*) q(ds'|s, a) \\ &= -e^{ru(s,a)} \int_S (V_1 Q)_{f^*} (V_2 Q)_{f^*} \cdots (V_{m-1} Q)_{f^*} \tilde{V}_m(f^*)(s') q(ds'|s, a) \end{aligned}$$

for all $(s, a) \in C$. Thus, by Definition 1, the definition of $A^*(s)$ and the fact that $f^*(s) \in A^*(s)$ for each $s \in S$, we deduce that

$$\begin{aligned} &\max_{a \in A(s)} p_m^r(s, \psi^*)(a) \\ &= \max_{a \in A(s)} \left(-e^{ru(s,a)} \int_S (V_1 Q)_{f^*} (V_2 Q)_{f^*} \cdots (V_{m-1} Q)_{f^*} \tilde{V}_m(f^*)(s') q(ds'|s, a) \right) \\ &= -e^{ru(s, f^*(s))} \int_S (V_1 Q)_{f^*} (V_2 Q)_{f^*} \cdots (V_{m-1} Q)_{f^*} \tilde{V}_m(f^*)(s') q(ds'|s, f^*(s)) \end{aligned}$$

for all $s \in S$. The last equality completes the proof for $r < 0$.

Assume now that $r = 0$ and define functions $w_1, \dots, w_m$ as follows

$$\begin{aligned} w_m(s, a) &:= v_m(s, a), \quad w_{m-1}(s, a) := v_{m-1}(s, a) + \int_S w_m(s', \psi^*) q(ds'|s, a), \\ w_{m-2}(s, a) &:= v_{m-2}(s, a) + \int_S w_{m-1}(s', \psi^*) q(ds'|s, a) \\ &= v_{m-2}(s, a) + \int_S v_{m-1}(s', \psi^*) q(ds'|s, a) \\ &\quad + \int_S [Q_{\psi^*} \tilde{v}_m(\psi^*)](s') q(ds'|s, a), \\ &\vdots \\ w_1(s, a) &:= v_1(s, a) + \int_S w_2(s', \psi^*) q(ds'|s, a) \\ &= v_1(s, a) + \int_S v_2(s', \psi^*) q(ds'|s, a) + \int_S [Q_{\psi^*} \tilde{v}_3(\psi^*)](s') q(ds'|s, a) + \cdots \\ &\quad + \int_S [Q_{\psi^*}^{(m-2)} \tilde{v}_m(\psi^*)](s') q(ds'|s, a). \end{aligned}$$

Note that by (14), we have that $w_1(s, \psi^*) = J_m^0(\psi^*)(s)$, $s \in S$. The proof proceeds along similar lines as for the risk averse generations. Therefore, analogously, we set

$$A^*(s) = \arg \max_{a \in A(s)} p_m^0(s, \psi^*)(a)$$

with $p_m^0(s, \psi^*)(\cdot)$ defined in (16). Clearly, $A^*(s) \neq \emptyset$ and $\psi^*(A^*(s)|s) = 1$, for each $s \in S$. Applying (39) to function $w_m, w_{m-1}, \dots, w_1$ in turn, we finally arrive at the following equality

$$\int_S w_1(s', \psi^*) q(ds'|s, a) = \int_S \left(v_1(s', f^*(s')) + \sum_{j=2}^m [\mathcal{Q}_{f^*}^{(j-1)} \tilde{v}_j(f^*)](s') \right) q(ds'|s, a),$$

$$(s, a) \in C.$$

Hence,

$$\begin{aligned} p_m^0(s, \psi^*)(a) &= u(s, a) + \int_S w_1(s', \psi^*) q(ds'|s, a) \\ &= u(s, a) + \int_S \left(v_1(s', f^*(s')) + \sum_{j=2}^m [\mathcal{Q}_{f^*}^{(j-1)} \tilde{v}_j(f^*)](s') \right) q(ds'|s, a) \end{aligned}$$

for all $(s, a) \in C$. Furthermore, Definition 1, the definition of $A^*(s)$ and the fact that $f^*(s) \in A^*(s)$ for each $s \in S$ yield the following

$$\begin{aligned} \max_{a \in A(s)} p_m^0(s, \psi^*)(a) &= \max_{a \in A(s)} \left(u(s, a) + \int_S \left(v_1(s', f^*(s')) + \sum_{j=2}^m [\mathcal{Q}_{f^*}^{(j-1)} \tilde{v}_j(f^*)](s') \right) q(ds'|s, a) \right) \\ &= u(s, f^*(s)) + \sum_{j=1}^m [\mathcal{Q}_{f^*}^{(j)} \tilde{v}_j(f^*)](s) \end{aligned}$$

for all $s \in S$. The last display completes the proof for $r = 0$.

Proof of Theorem 3. Let $v_j = \beta^j u$, $j \in T$ and let us now consider $r = 0$. We have that $J_\infty^0(\psi)(s) = \lim_{m \rightarrow \infty} J_m^0(\psi)(s)$, for all $\psi \in \Psi$ and $s \in S$. Note that by (21), $J_\infty^0(\psi) = \beta \hat{J}_\infty^0(\psi)$. If u satisfies (A9), from (32), it follows that the sequence $\{J_m^0(\psi)(s)\}$ converges to $J_\infty^0(\psi)(s)$ uniformly in $\psi \in \Psi$ and $s \in S$ as $m \rightarrow \infty$. This fact and Lemma 3 imply that $(\{J_\infty^0(\psi_n)\}, J_\infty^0(\psi)) \in \Gamma$. The proof now proceeds as that of Theorem 1 with $J_m^0(\psi)$ replaced by $J_\infty^0(\psi)$.

For $r < 0$ we also have the uniform convergence of $J_m^r(\psi)(s)$ to $J_\infty^r(\psi)(s)$ over $\psi \in \Psi$ and $s \in S$, which quite easily follows from our boundedness assumption on u . Therefore, the arguments used in the proof of Theorem 1 can also be applied to $J_\infty^r(\psi)$ instead of $J_\infty^0(\psi)$. $\square$

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